A CATEGORICAL APPROACH TO PROBABILITY THEORY

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The aim of this paper is to give a categorical definition of random processes and provide tools for their study.

A process is meant to describe something evolving in time, the history before time t «probabilistically» determining what will happen later on. For instance, it may represent a moving point x, being at time t in a space Ω_t endowed with a σ-algebra $\mathcal{B}_t$; the problem is then the exact position of x in Ω_t.

In the very particular example of a Markov process, time is running through N and the position of x in Ω_{n+1} only depends on where it was in Ω_n. So, for each n, a map f_n from $\Omega_n \times \mathcal{B}_{n+1}$ to $[0,1]$ is given: $f_n(\omega_n, B_{n+1})$ is the probability for x to be in B_{n+1} at time $n+1$ if it was on ω_n at time n. f_n is thus asked to satisfy the two following properties: for each ω_n, $f_n(\omega_n, .)$ is a probability measure on $(\Omega_{n+1}, \mathcal{B}_{n+1})$, and for each B_{n+1}, $f_n(., B_{n+1})$ is measurable. f_n is called a transition probability [4], or a probabilistic mapping [3] from $(\Omega_n, \mathcal{B}_n)$ to $(\Omega_{n+1}, \mathcal{B}_{n+1})$. The process is then entirely defined by the f_n's.

But if time runs through R, we need a transition probability f_t^s from $(\Omega_s, \mathcal{B}_s)$ to $(\Omega_t, \mathcal{B}_t)$ for each couple (s, t) with $s \leq t$. Then, if $r \leq s \leq t$, there are two ways of computing the probability for x to be in B_t knowing it was on ω_r at time r: forgetting s, which gives $f_t^r(\omega_r, B_t)$; or considering how x behaved at time s, it seems then reasonable to take the mean value of $f_t^s(\omega_s, B_t)$ (for ω_s running through Ω_s) relatively to the probability measure $f_s^r(\omega_r, .)$ on Ω_s, which yields to $\int f_t^s(., B_t)\, d f_s^r(\omega_r, .)$. This integral is shown to define a transition probability from Ω_r to Ω_t, called the composite of f_s^r and f_t^s, which is asked, in «good» processes, to be the same as f_t^r. This equality is called the Chapman-Kolmogoroff relation.

The composition of transition probabilities is associative. This property is equivalent to Fubini Theorem for bounded functions and its proof, as well as that of stability of transition probabilities for this law, is rather technical. These results will be consequences of the following one: the transition probabilities and their composition form the Kleisli category of a monad on the category of measurable spaces. A similar monad will be constructed on a category of topological spaces (generalizing the one defined by Swirszcz [6] on compact spaces).

As F. W. Lawvere already pointed out in an unpublished paper [3] in 1962, most problems in probability and statistics theory can be translated in terms of diagrams in these Kleisli categories.

In the sequel we'll mainly study projective limits which will lead us to construct probability measures on sample sets of processes.

I'd like to express my gratitude to the organizers of the International Conference on Categorical Aspects of Topology and Analysis, who gave me the opportunity to give a lecture and to publish this work, and to warmly thank Andrée Charles Ehresmann, whose constant help and understanding made it possible.

I. THE PROBABILITY MONADS.

1. Notations. $\mathcal{M}es$ is the category of measurable spaces; an object will be denoted by Ω and its σ-algebra by $\mathcal{B}_\Omega$. The morphisms are the measurable maps. $\mathcal{P}ol$ is the category of Polish spaces (topological spaces underlying a complete metric space); again, an object is called Ω and $\mathcal{B}_\Omega$ is its Borel σ-algebra. The morphisms are the continuous maps.

A monad is going to be constructed on both $\mathcal{M}es$ and $\mathcal{P}ol$; the definitions being very much the same, in the sequel $\mathcal{H}$ will stand for either of them, unless otherwise notified.

2. Construction.

a) The functor Π (called P in [3]): If Ω is an object of $\mathcal{H}$, $\Pi(\Omega)$ is the set of probability measures on Ω (i.e., the σ-additive maps from $\mathcal{B}_\Omega$ to $[0,1]$ sending Ω to 1), endowed:

. if $\mathcal{H} = \mathcal{M}es$, with the initial σ-algebra for the following evaluation maps, where B runs through $\mathcal{B}_\Omega$:

$$p_B : \Pi(\Omega) \to [0,1] : P \mapsto P(B);$$

. if $\mathcal{H} = \mathcal{P}ol$, with the initial topology for the maps

$$\xi_f : \Pi(\Omega) \to \mathrm{R} : \quad P \mapsto \int f \, dP,$$

where f is any bounded continuous map from Ω to R; $\Pi(\Omega)$ is then a Polish space (its topology is called the weak topology) [5].

If $f : \Omega \to \Omega'$ is a morphism of $\mathcal{H}$, and P is in $\Pi(\Omega)$, the probability measure on Ω' image of P by f is defined by

$$\Pi(f)(P)(B') = P(\overset{-1}{f}(B')) \quad \text{for every } B' \text{ in } \mathcal{B}_{\Omega'}.$$

b) The natural transform $\eta : Id_{\mathcal{H}} \Rightarrow \Pi$: The characteristic function of an element B of $\mathcal{B}_\Omega$ is denoted by χ_B. For ω in Ω, the probability measure concentrated on ω is defined by

$$\eta_\Omega(\omega)(B) = \chi_B(\omega), \quad \text{for each } B \text{ in } \mathcal{B}_\Omega.$$

c) The natural transform $\mu : \Pi^2 \Rightarrow \Pi$: For P' in $\Pi^2(\Omega)$, a probability measure on Ω is defined by:

$$\mu_\Omega(P')(B) = \int p_B \, dP', \quad \text{for every } B \text{ in } \mathcal{B}_\Omega.$$

These integrals are well-defined for each p_B is measurable from $\Pi(\Omega)$ to $[0,1]$, hence is integrable for P'. The measurability of p_B if Ω is an object of $\mathcal{P}ol$ follows from the fact that, Ω being metrizable, we have $\mathcal{B}_\Omega = \bigcup_{a<A} \mathcal{G}_a$, where A is the first uncountable ordinal and: $\mathcal{G}_0$ is the set of open sets of Ω,

$$\mathcal{G}_{\alpha+1} \begin{cases} = \{ \bigcap_n \downarrow B_n \mid B_n \in \mathcal{G}_\alpha \} & \text{if } \alpha \text{ is even,} \\ = \{ \bigcup_n \uparrow B_n \mid B_n \in \mathcal{G}_\alpha \} & \text{if } \alpha \text{ is odd,} \end{cases}$$

$$\mathcal{G}_\alpha = \{ \bigcup_n \uparrow B_n \mid B_n \in \bigcup_{\beta<\alpha} \mathcal{G}_\beta \} \quad \text{if } \alpha \text{ is limit,}$$

and is proved by induction on the class of B.

The σ-additivity of $\mu_\Omega(P')$ is a consequence of the monotone convergence theorem.

3. Theorem 1. *(Π, η, μ) is a monad on $\mathcal{H}$.*

Δ. First let us prove the following properties, valid for any morphism $f: \Omega \to \Omega'$ of $\mathcal{H}$, any P in $\Pi(\Omega)$, P' in $\Pi^2(\Omega)$, ω in Ω, $\theta: \Omega \to \mathrm{R}$ and $\theta': \Omega' \to \mathrm{R}$ bounded measurable functions:

a) $\int \theta' d\Pi(f)(P) = \int \theta' \circ f \, dP$,

b) $\int \theta \, d\eta_\Omega(\omega) = \theta(\omega)$,

c) if ξ_θ is defined by $\xi_\theta(P) = \int \theta \, dP$, ξ_θ is measurable from $\Pi(\Omega)$ to R,

d) $\int \theta \, d\mu_\Omega(P') = \int \xi_\theta \, dP'$.

In the case where θ is of the form χ_B, this follows from the definitions ; by linearity of $\int$ this is still true if θ is a simple function. The general case is a consequence of the monotone convergence theorem and the fact that θ is the increasing pointwise limit of a sequence of simple functions.

If $f: \Omega \to \Omega'$ is in $\mathcal{H}$, so is $\Pi(f)$: When $\mathcal{H} = \mathcal{M}es$, measurability of $\Pi(f)$ is obvious; when $\mathcal{H} = \mathcal{P}ol$, continuity of $\Pi(f)$ follows from formula a. Now Π is a functor because, if $g: \Omega' \to \Omega''$ is another morphism of $\mathcal{H}$,

$$(g \circ f)^{-1}(B'') = f^{-1}(g^{-1}(B'')) \text{ for } B'' \in \mathcal{B}_{\Omega''}.$$

η_Ω is clearly in $\mathcal{H}$ (by definition in $\mathcal{M}es$ and by formula b in $\mathcal{P}ol$). So is μ_Ω:

- if $\mathcal{H} = \mathcal{M}es$, this will follow from property c above, applied to $\Pi(\Omega)$, $\theta = p_B$ and P';
- if $\mathcal{H} = \mathcal{P}ol$, it is a consequence of formula d.

The diagramms

$$\begin{array}{ccc} \Omega & \xrightarrow{\quad f \quad} & \Omega' \\ \downarrow{\scriptstyle \eta_\Omega} & & \downarrow{\scriptstyle \eta_{\Omega'}} \\ \Pi(\Omega) & \xrightarrow{\quad \Pi(f) \quad} & \Pi(\Omega') \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Pi^2(\Omega) & \xrightarrow{\quad \Pi^2(f) \quad} & \Pi^2(\Omega') \\ \downarrow{\scriptstyle \mu_\Omega} & & \downarrow{\scriptstyle \mu_{\Omega'}} \\ \Pi(\Omega) & \xrightarrow{\quad \Pi(f) \quad} & \Pi(\Omega') \end{array}$$

commute: Only the second one requires some work: if P' is in $\Pi^2(\Omega)$ and B' in $\mathcal{B}_{\Omega'}$,

$$(\Pi(f) \circ \mu_\Omega(P'))(B') = \mu_\Omega(P')(f^{-1}(B')) = \int p_{f^{-1}(B')} dP'$$

and

$$(\mu_{\Omega'} \circ \Pi^2(f)(P'))(B') = \int p_{B'} d\Pi^2(f)(P') = \int p_{B'} \circ \Pi(f) \, dP'$$

from a. But since $p_{f^{-1}(B')} = p_{B'} \circ \Pi(f)$, the commutativity follows.

Unitarity of η is easy. Let's prove the associativity of μ. If P'' is in $\Pi^3(\Omega)$ and B is in $\mathcal{B}_\Omega$,

$$(\mu_\Omega \circ \Pi(\mu_\Omega)(P''))(B) = \int p_B \circ \mu_\Omega \, dP'' = \int \xi_{p_B} \, dP''$$

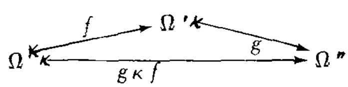
from a and definition of μ, and

$$(\mu_\Omega \circ \mu_{\Pi(\Omega)}(P''))(B) = \int p_B \, d\mu_{\Pi(\Omega)}(P'').$$

Equality follows from d. Δ

4. The Kleisli category of (Π, η, μ).

The Kleisli category associated to (Π, η, μ) is called $\mathcal{PT}$. If f, g are morphisms in $\mathcal{PT}$, we write



The canonical functor from $\mathcal{H}$ to $\mathcal{PT}$ sends

$$\Omega \xrightarrow{\ h\ } \Omega' \quad \text{to} \quad \Omega \overset{\overset{\kappa}{h}}{\longleftrightarrow} \Omega', \text{ where } \overset{\kappa}{h} = \eta_{\Omega'} \circ h.$$

A Kleisli morphism $f : \Omega \leftrightarrow \Omega'$ is a transition probability in the following way: let us define

$$F : \Omega \times \mathcal{B}_{\Omega'} \to [0, 1] \quad \text{by} \quad F(\omega, B') = f(\omega)(B').$$

Then $F(\omega, .)$ is a probability measure on Ω' and $F(., B') = p_{B'} \circ f$ is measurable, so F is a transition probability.

If $\mathcal{H} = \mathcal{M}es$, the transformation $f \mapsto F$ is a one-to-one correspondence between $\mathcal{PT}$ and the class of transition probabilities.

Moreover, the composition in $\mathcal{PT}$ is the usual composition of transition probabilities, since:

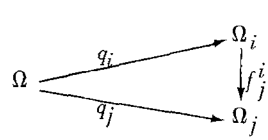
$$(g \kappa f)(\omega)(B) = \int p_B \circ g \, df(\omega) = \int g(.)(B) \, df(\omega)$$
$$= \int G(., B) \, dF(\omega, .)$$

with the above notations. So, Chapman-Kolmogoroff relation means that a «good» process (cf. the introduction) is defined by a functor from the ordered set R to $\mathcal{PT}$.

II. PROBLEMS OF PROJECTIVE LIMITS IN $\mathcal{PT}$.

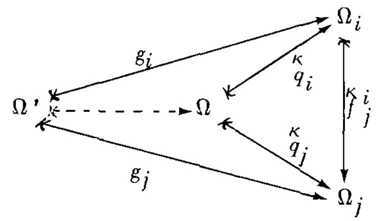
In the study of a process, an important problem is to find a probability measure on the sample set, compatible with the given transition probabilities. This sample set being the projective limit of the sets of «histories before a time t», the existence of a limit for a projective system of probability measures soon arises, which can be translated in terms of preservation by $\overset{\kappa}{\ }$ of projective limits in $\mathcal{H}$ (which still represents $\mathcal{M}es$ or $\mathcal{P}ol$). Let us recall that $\mathcal{M}es$ admits all limits and $\mathcal{P}ol$ all countable limits.

In the sequel, we consider the following situation: Ω is the projective limit of a functor F from a filtered set $(I, \geq)$ to $\mathcal{H}$, and is characterized by the commutative diagrams, for $i \geq j$:


;

we want to know how $\overset{\kappa}{\cdot}$ preserves the limit, that is: when and how can the dotted arrow of the following diagram be filled up?

(D)



Notice that commutativity of the outer diagram means that $\Pi(f^i_j)(g_i(\omega')) = g_j(\omega')$ for each ω', so that the families $(g_i(\omega'))_i$ are projective systems of probability measures. Hence, the first problem will be the existence of a limit $g(\omega')$ for such a system, and the second one the fact that the so defined g is a morphism of $\mathcal{PT}$. The results will be of a different nature in $\mathcal{M}es$ and in $\mathcal{P}ol$.

1. Theorem 1. *If* $\mathcal{H} = \mathcal{M}es$ *and the* q_i*'s are onto:* $(\overset{\kappa}{q}_i)_{i\in I}$ *is universal among the cones* $(g_i)_{i\in I}$ *with basis* $\overset{\kappa}{\cdot}F$ *such that:*

(*) For all increasing sequence $(i_n)_n$ *of* I *and each* $(B_{i_n})_n$ *in* $\prod_n \mathcal{B}_{\Omega_{i_n}}$ *such that* $\cap\downarrow q^{-1}_{i_n}(B_{i_n}) = \emptyset$, *the sequence* $p_{B_{i_n}} \circ g_{i_n}$ *pointwisely converges to* 0.

Δ. Let ω' be in Ω'. $(g_i(\omega'))_i$ being a projective system of probability measures one can define an application $g(\omega')$ from the algebra $\mathcal{B}' = \bigcup_{i\in I} q_i^{-1}(\mathcal{B}_{\Omega_i})$ generating $\mathcal{B}_\Omega$ to $[0,1]$ by

$$g(\omega')(q_i^{-1}(B_i)) = g_i(\omega')(B_i).$$

Thanks to condition (*), $g(\omega')$ is a probability measure on $\mathcal{B}'$ and can subsequently be uniquely extended to $\mathcal{B}_\Omega$ (cf. [4]). It remains to show that g is measurable from Ω' to $\Pi(\Omega)$, which is equivalent to the measurability of all the $p_B \circ g$ for B in $\mathcal{B}_\Omega$. But the set of B's for which $p_B \circ g$ is measurable contains $\mathcal{B}'$ and is a monotone class, so that it's $\mathcal{B}$ itself. Δ

Remark. Condition (*) was necessary for the existence of the $g(\omega')$'s. Measurability of g did not require any further hypothesis.

2. Theorem 1 bis. *If* $\mathcal{H} = \mathcal{P}ol$, $(I,\geq) = (\mathrm{N},\geq)$ *and the* q_n *'s are onto,* Ω *is the projective limit of* $\overset{\kappa}{\cdot}F$.

Δ. The Ω_n's and Ω being Polish, a probability measure in the sense we use is also a measure in the Bourbaki sense. Hence all the systems $(g_n(\omega'))_{n\in\mathrm{N}}$ have a limit $g(\omega')$ [1]. (This would be true even if I had only a cofinal countable subset.) Now we must show that g is continuous from Ω' to $\Pi(\Omega)$. We'll use the following result [5]:

A subset X of $\Pi(\Omega)$ is relatively compact iff it is uniformly tight, which means that, for all $\epsilon > 0$ there is a compact K_ϵ in Ω such that $P(K_\epsilon) > 1-\epsilon$ for all P in X.

Let $(\omega'_n)_n$ converge to ω' in Ω': we only need to show that, for any continuous map θ to R, bounded by 1, $\int \theta \, dg(\omega'_n)$ converge to $\int \theta \, dg(\omega')$. We'll use the following notations:

$$g_p(\omega'_n) = P^p_n, \quad g_p(\omega') = P^p, \quad g(\omega'_n) = P_n \quad \text{and} \quad g(\omega') = P.$$

1° We get that the set $\{ P^p_n \mid n \in \mathrm{N} \}$ *is uniformly tight* since it is the image by the continuous map g_p of the relatively compact set $\{ \omega'_n \mid n \in \mathrm{N} \}$.

2° *The set* $\{ P_n \mid n \in \mathrm{N} \}$ *is also uniformly tight*: let's fix ϵ in R^*_+. There is a compact subset K_1 of Ω_1 such that $P^1_n(K_1) > 1 - \frac{\epsilon}{2}$ for each n. Suppose $K_1, \dots, K_p$ are constructed such that, for each $k \leq p$, K_k is a compact subset of Ω_k, contained in $(f^k_{k-1})^{-1}(K_k)$ (for $k > 1$) and satisfying:

$$P^k_n(K_k) > 1 - \left(\frac{\epsilon}{2} + \dots + \frac{\epsilon}{2^k}\right)$$

for each n; there is a compact K'_{p+1} in Ω_{p+1} with

$$\inf_n P^{p+1}_n(K'_{p+1}) > 1 - \frac{\epsilon}{2^{p+1}}.$$

The compact $K_{p+1} = K'_{p+1} \cap (f^{p+1}_p)^{-1}(K_p)$ satisfies

$$\inf_n P^{p+1}_n(K_{p+1}) > 1 - \left(\frac{\epsilon}{2} + \dots + \frac{\epsilon}{2^{p+1}}\right).$$

So a sequence $(K_p)_p$ can be inductively constructed, such that K_p is a compact subset of Ω_p contained in $(f^p_{p-1})^{-1}(K_{p-1})$ satisfying $\inf_n P^p_n(K_p) > 1 - \epsilon$. The (decreasing) intersection K of the $q_p^{-1}(K_p)$ is such that $\inf_n P_n(K) > 1 - \epsilon$. It is enough now to prove that K is compact: if $\mathfrak{U}$ is an ultrafilter on K, $q_p(\mathfrak{U})$ is an ultrafilter on K_p (since q_p is onto) and hence converges to a ω_p in K_p. The f^p_q being continuous

$$f^p_q(\omega_p) = \omega_q \quad \text{for each } p \geq q;$$

therefore there is an ω in Ω satisfying $q_p(\omega) = \omega_p$ for each p, which means that ω is in fact in K. As Ω is the topological projective limit of the Ω_p's, it's then easy to prove that $\mathfrak{U}$ converges to this ω; compactness of K follows.

3° Let $A = \{ \theta_p \circ q_p \mid p \in \mathrm{N}, \ \theta_p : \Omega_p \to \mathrm{R} \text{ continuous bounded} \}$. A is a subalgebra of the algebra of continuous bounded maps from Ω to R; it contains the constant maps and it separates the points of Ω; so Stone-Weierstrass Theorem ensures that A is dense in this algebra, endowed with the topology of uniform convergence on compact subsets.

4° We are now able to prove the convergence of $(\int \theta \, dP_n)_n$ to $\int \theta \, dP$. Let $\epsilon > 0$ be fixed; by 2 there exists a compact subset K of Ω such that

$$\inf(\{ P_n(K) \mid n \in \mathrm{N} \} \cup \{ P(K) \}) > 1 - \frac{\epsilon}{8},$$

hence

$$(1) \qquad \left| \int \theta \, dP_n - \int \theta \, dP \right| < \left| \int_K \theta \, dP_n - \int_K \theta \, dP \right| + \frac{\epsilon}{4}.$$

By 3, there exists a continuous map $\theta_p : \Omega_p \to \mathrm{R}$ (that can be chosen bounded by 1) such that: $\sup_K | \theta(\omega) - \theta_p \circ q_p(\omega) | < \frac{\epsilon}{8}$; it follows:

$$(2)\quad \left| \int_K \theta \, dP_n - \int_K \theta \, dP \right| \le \left| \int_K \theta_p \circ q_p \, dP_n - \int_K \theta_p \circ q_p \, dP \right| + \frac{\epsilon}{4}$$
$$\le \left| \int \theta_p \circ q_p \, dP_n - \int \theta_p \circ q_p \, dP \right| + \frac{\epsilon}{4} + \frac{\epsilon}{4}.$$

But since

$$P_n^p = \Pi(q_p)(P_n) \quad \text{and} \quad P^p = \Pi(q_p)(P),$$

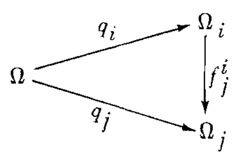
the second member above is

$$\left| \int \theta_p \, dP_n^p - \int \theta_p \, dP^p \right| + \frac{\epsilon}{2},$$

so that, since P_n^p converges to P^p, it is less than $\frac{3\epsilon}{4}$ if n is greater than a N in N. Hence, for $n \ge N$, $\left| \int \theta \, dP_n - \int \theta \, dP \right|$ is less than ϵ, from inequalities (1) and (2). So g is a morphism of $\mathcal{P}o\ell$ filling up diagram (D), and the proof is complete. Δ

3. A general process on discrete time is described by a sequence (f_n) of transition probabilities, f_n being from the set of «histories before n» $\Omega_1 \times \ldots \times \Omega_n$ to Ω_{n+1}. We'll see that this sequence induces a functor from $(\mathrm{N}, \le)$ to $\mathcal{P}\mathcal{T}$ and, using the above theorems, we'll construct a transition probability from each $\Omega_1 \times \ldots \times \Omega_p$ to the sample set $\prod_n \Omega_n$, compatible with the f_n's. This result, which contains Ionescu Tulcea Theorem [4], is a corollary of Theorem 3 below, itself a consequence of the following

Theorem 2. *Here $\mathcal{H} = \mathcal{M}es$. If $(I, \ge)$ is a filtered ordered set and F a functor $(I, \ge) \to \mathcal{M}es$ whose projective limit is given by the commutative diagrams*



$i \ge j$

with the further property

(sm) For any increasing sequence $(i_n)_n$ in I and any $(\omega_{i_n})_n$ in $\prod_n \Omega_{i_n}$ such that: $f_{i_n}^{i_{n+1}}(\omega_{i_{n+1}}) = \omega_{i_n}$, there is a ω in Ω satisfying $q_{i_n}(\omega) = \omega_{i_n}$,

and if G is a functor $(I, \le) \to \mathcal{P}\mathcal{T}$ verifying:

a) For $i \ge j$, $G(i,j) = g_j^i : \Omega_j \leftarrow\!\!\!\!\kappa\!\!\rightarrow \Omega_i$ is left inverse to $\overset{\kappa}{f}{}_j^i$,

b) For $i \ge i_0$, $\omega_{i_0} \in \Omega_{i_0}$, $B_i \in \mathcal{B}_{\Omega_i}$, $\omega_{i_0} \notin f_{i_0}^i(B_i)$ implies $g_{i_0}^i(\omega_{i_0})(B_i) = 0$,

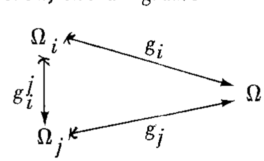
then for each i_0 in I there is a unique morphism $g_{i_0} : \Omega_{i_0} \leftarrow\!\!\!\!\kappa\!\!\rightarrow \Omega$ such that

$$\overset{\kappa}{q}_i \,\kappa\, g_{i_0} = g_{i_0}^i \quad \textit{for every } i \ge i_0$$

and

$$g_{i_0}(\omega_{i_0})(B) = 0 \quad \textit{for every } (\omega_{i_0}, B) \textit{ in } \Omega_{i_0} \times \mathcal{B}_\Omega \textit{ with } \omega_{i_0} \notin q_{i_0}(B).$$

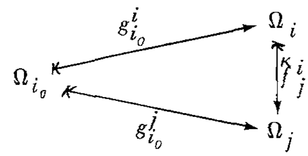
Moreover, if I is totally ordered, the diagrams



$j \ge i$

commute.

Δ. Let us first notice that property a for G means that



commutes for $i \geq j \geq i_0$. This, together with the fact that Ω is also the projective limit of the restriction of F to $\{ i \in I \mid i \geq i_0 \}$, implies that the existence and unicity of g_{i_0} will be proved, using Theorem 1, as soon as we have shown that $(g^i_{i_0})_{i \geq i_0}$ has property (*) of this theorem. Indeed, if it has not, there is a ω_{i_0} in Ω_{i_0}, an increasing sequence $(i_n)_n$ in I and sets B_{i_n} in $\mathcal{B}_{\Omega_{i_n}}$ such that:

$$\bigcap_n \downarrow q^{-1}_{i_n}(B_{i_n}) = \emptyset \quad \text{and} \quad \lim_n g^{i_n}_{i_0}(\omega_{i_0})(B_{i_n}) > 0 .$$

Suppose we have constructed $(\omega_{i_0}, \dots, \omega_{i_p})$ in $\Omega_{i_0} \times \dots \times \Omega_{i_p}$ satisfying:

$$\text{For } q \geq 1,\ f^{i_q}_{i_{q-1}}(\omega_{i_q}) = \omega_{i_{q-1}} \quad \text{and} \quad \lim_n g^{i_n}_{i_q}(\omega_{i_q})(B_{i_n}) > 0 .$$

Since $g^{i_n}_{i_p} = g^{i_n}_{i_{p+1}} \kappa\, g^{i_{p+1}}_{i_p}$ for $n \geq p+1$, we have

$$g^{i_n}_{i_p}(\omega_{i_p})(B_{i_n}) = \int p_{B_{i_n}} \circ g^{i_n}_{i_{p+1}} \, d g^{i_{p+1}}_{i_p}(\omega_{i_p}).$$

Compatibility of integral and pointwise increasing limit implies that

$$\int \lim_n p_{B_{i_n}} \circ g^{i_n}_{i_{p+1}} \, d g^{i_{p+1}}_{i_p}(\omega_{i_p}) > 0,$$

and hence that

$$g^{i_{p+1}}_{i_p}(\omega_{i_p})(\{ \omega_{i_{p+1}} \in \Omega_{i_{p+1}} \mid \lim_n g^{i_n}_{i_{p+1}}(\omega_{i_{p+1}})(B_{i_n}) > 0 \}) > 0 .$$

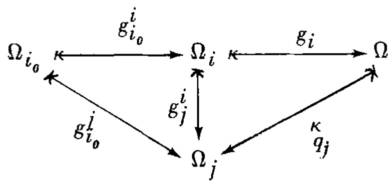
Condition b then gives a $\omega_{i_{p+1}}$ in $\Omega_{i_{p+1}}$ such that

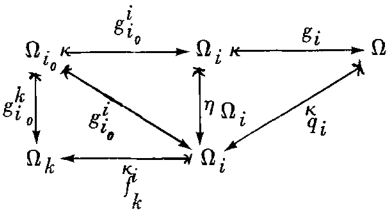
$$f^{i_{p+1}}_{i_p}(\omega_{i_{p+1}}) = \omega_{i_p} \quad \text{and} \quad \lim_n g^{i_n}_{i_{p+1}}(\omega_{i_{p+1}})(B_{i_n}) > 0 .$$

The sequence $(\omega_{i_n})_n$ we've just constructed inductively satisfies $f^{i_{n+1}}_{i_n}(\omega_{i_{n+1}}) = \omega_{i_n}$ for each n ; so condition (sm) provides us with a ω in Ω such that $q_{i_n}(\omega) = \omega_{i_n}$ for every n. Now since $g^{i_n}_{i_n}(\omega_{i_n})(B_{i_n}) > 0$, ω_{i_n} is in B_{i_n} and ω in $q^{-1}_{i_n}(B_{i_n})$. This is absurd for $\bigcap_n q^{-1}_{i_n}(B_{i_n})$ was supposed to be empty.

So we have our g_{i_0} ; to show that $g_{i_0}(\omega_{i_0})(B) = 0$ if $\omega_{i_0} \notin q_{i_0}(B)$, remark that the set of B for which this is true (for a fixed ω_{i_0}) is a monotone class containing the algebra $\bigcup_{i > i_0} q_i^{-1}(\mathcal{B}_{\Omega_i})$, which generates $\mathcal{B}_\Omega$.

It remains to consider the totally ordered case and to show that, for $i \geq i_0$, we have $g_{i_0} = g_i \kappa\, g^i_{i_0}$. Fix $i \geq i_0$. For $j \geq i \geq k \geq i_0$, the following diagrams commute:





Hence

$$\overset{\kappa}{q}_j \,\kappa\, (g_i \,\kappa\, g^i_{i_0}) = g^j_{i_0} \quad \text{and} \quad (\overset{\kappa}{f}{}^i_k \,\kappa\, \overset{\kappa}{q}_i)\,\kappa\,(g_i \,\kappa\, g^i_{i_0}) = \overset{\kappa}{q}_k \,\kappa\,(g_i \,\kappa\, g^i_{i_0}) = g^k_{i_0}\,,$$

so that for every $j \geq i$, $\overset{\kappa}{q}_j \,\kappa\, (g_i \kappa g^i_{i_0}) = g^j_{i_0}$; from uniqueness of g_{i_0}, we get the expected commutativity. Δ

4. Application to processes.

a) **Theorem 3.** *Let $(E_\alpha)_{\alpha\in I}$ be a family of objects of $\mathfrak{M}es$ indexed by a well-ordered set $(I,\geq)$, Ω_α (resp. Ω) the product of $(E_\beta)_{\beta\leq\alpha}$ (resp. $(E_\alpha)_{\alpha\in I}$). Given a family $(f_\alpha)_{\alpha\in I}$ where $f_\alpha : \Omega_\alpha \;\kappa\!\!\longrightarrow E_{\alpha+1}$, there is, for each α_0 in I, a unique $g_{\alpha_0} : \Omega_{\alpha_0} \;\kappa\!\!\longrightarrow \Omega$ such that:*

For $\omega_{\alpha_0} \in \Omega_{\alpha_0}$, $B_{\alpha_0} \in \mathcal{B}_{\Omega_{\alpha_0}}$, $(F_{\alpha_0+i})_{1\leq i\leq n} \in \prod_{1\geq i\geq n} \mathcal{B}_{\Omega_{\alpha_0+i}}$,

$$g_{\alpha_0}(\omega_{\alpha_0})(B) = \chi_{B_{\alpha_0}}(\omega_{\alpha_0}) \int_{F_{\alpha_0+1}} df_{\alpha_0}(\omega_{\alpha_0}) \int_{F_{\alpha_0+2}} df_{\alpha_0+1}(\omega_{\alpha_0}, x_{\alpha_0+1})\ldots$$

$$\ldots\int_{F_{\alpha_0+n}} df_{\alpha_0+n-1}(\omega_{\alpha_0}, x_{\alpha_0+1},\ldots, x_{\alpha_0+n-1})$$

where $B = B_{\alpha_0} \times \prod_{1\leq i\leq n} F_{\alpha_0+i} \times \prod_{\alpha>\alpha_0+n} E_\alpha$.

Δ. The projections from Ω to Ω_α and Ω_α to Ω_β if $\alpha\geq\beta$ are respectively denoted by q_α and q^α_β. The q_α's are onto and satisfy (sm).

a) For each β in I and ω_β in Ω_β, let us call $g^{\beta+1}_\beta(\omega_\beta)$ the probability measure on $\Omega_{\beta+1} = \Omega_\beta \times E_{\beta+1}$, product of $\eta_{\Omega_\beta}(\omega_\beta)$ and $f_\beta(\omega_\beta)$, given by

$$g^{\beta+1}_\beta(\omega_\beta)(B_\beta \times F_{\beta+1}) = \chi_{B_\beta}(\omega_\beta)\cdot f_\beta(\omega_\beta)(F_{\beta+1}) \quad \text{for} \begin{cases} B_\beta \in \mathcal{B}_{\Omega_\beta} \text{ and} \\ F_{\beta+1} \in \mathcal{B}_{E_{\beta+1}}. \end{cases}$$

$g^{\beta+1}_\beta$ is measurable: the set $\{ B \in \mathcal{B}_\Omega \mid p_B \circ g^{\beta+1}_\beta$ is measurable $\}$ is a monotone class and contains the disjoint unions of the sets $B_\beta \times F_{\beta+1}$ which form an algebra generating $\mathcal{B}_\Omega$; so it's $\mathcal{B}_\Omega$ itself.

The same techniques prove that $g^{\beta+1}_\beta(\omega_\beta)(B_{\beta+1}) = 0$ if $\omega_\beta \notin q^{\beta+1}_\beta(B_{\beta+1})$.

b) Let us consider the couples (J, G_J), where J is a beginning section of I and G_J a functor $(J,\leq) \to \mathcal{PT}$, such that $G_J(\beta,\alpha) = h^\alpha_\beta : \Omega_\beta \;\kappa\!\!\longrightarrow \Omega_\alpha$ if $\beta\leq\alpha$ satisfies:

(i) h^α_β is left inverse to $\overset{\kappa}{q}{}^\alpha_\beta$.

(ii) $h^{\beta+1}_\beta = g^{\beta+1}_\beta$.

(iii) $h^\alpha_\beta(\omega_\beta)(B_\alpha) = 0$ if $\omega_\beta \notin q^\alpha_\beta(B_\alpha)$, for $\omega_\beta \in \Omega_\beta$ and $B_\alpha \in \mathcal{B}_{\Omega_\alpha}$.

The obvious order on these couples is inductive and, from Zorn's Lemma, there is a maximal element (I_1, G_{I_1}). Let's denote $G_{I_1}(\alpha,\beta)$ by g^{α}_{β} $(\alpha \geq \beta)$ and prove that I_1 is I itself. If I_1 is strictly included in I, then $\alpha = \inf\{\gamma \in I \mid \gamma \notin I_1\}$ exists. I_2 will be the set obtained by adding α to I_1. We'll consider the two possible cases:

1. If α has a predecessor α', let's denote

$$g^{\alpha}_{\gamma} = g^{\alpha}_{\alpha'} \kappa\, g^{\alpha'}_{\gamma} \quad \text{for each } \gamma < \alpha, \qquad g^{\alpha}_{\alpha} = \eta_{\Omega_\alpha}$$

and denote by G_{I_2} the map sending $(\beta,\gamma) \in I_2^2$, $\beta < \gamma$ to g^{γ}_{β}. It's then easy to prove that G_{I_2} is a functor $(I_2, \leq) \to \mathcal{P}\mathcal{T}$ and that g^{γ}_{β} is left inverse to $\overset{\kappa}{q}{}^{\gamma}_{\beta}$ for $\beta \leq \gamma$ in I_2. Now we want to show that $g^{\alpha}_{\beta}(\omega_\beta)(B_\alpha) = 0$ if $\omega_\beta \notin q^{\alpha}_{\beta}(B_\alpha)$: We know it is true if $\beta = \alpha'$. If $\beta \leq \alpha'$ and $\omega_\beta \notin q^{\alpha}_{\beta}(B_\alpha)$,

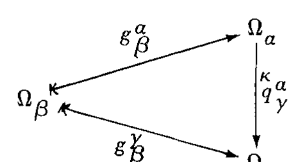
$$g^{\alpha}_{\beta}(\omega_\beta)(B_\alpha) = (g^{\alpha}_{\alpha'} \kappa\, g^{\alpha'}_{\beta})(\omega_\beta)(B_\alpha) = \int g^{\alpha}_{\alpha'}(\cdot)(B_\alpha)\, d g^{\alpha'}_{\beta}(\omega_\beta).$$

The set

$$X_{\alpha'} = \{\omega_{\alpha'} \in \Omega_{\alpha'} \mid g^{\alpha}_{\alpha'}(\omega_{\alpha'})(B_\alpha) \neq 0\}$$

is included in $q^{\alpha}_{\alpha'}(B_\alpha)$, by definition of $g^{\alpha}_{\alpha'} = g^{\alpha'+1}_{\alpha'}$. So $q^{\alpha'}_{\beta}(X_{\alpha'}) \subset q^{\alpha}_{\beta}(B_\alpha)$. Hence, if $\omega_\beta \notin q^{\alpha}_{\beta}(B_\alpha)$, $\omega_\beta \notin q^{\alpha'}_{\beta}(X_{\alpha'})$ and from (iii), $g^{\alpha'}_{\beta}(\omega_\beta)(X_{\alpha'}) = 0$. It follows that the above integral is zero.

2. If α is a limit ordinal, Ω_α is the projective limit of the $(\Omega_\beta)_{\beta<\alpha}$ and conditions (i), (ii), (iii) satisfied by G_{I_1} allow us to use Theorem 2; for each $\beta < \alpha$, there is a unique g^{α}_{β} such that the diagram



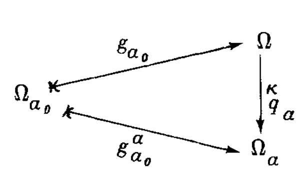
commute for $\beta \leq \gamma \leq \alpha$. If $\beta = \gamma$, we get that g^{α}_{β} is left inverse to $\overset{\kappa}{q}{}^{\alpha}_{\beta}$. Moreover,

$$g^{\alpha}_{\beta}(\omega_\beta)(B_\alpha) = 0 \quad \text{if} \quad \omega_\beta \notin q^{\alpha}_{\beta}(B_\alpha),$$

and, I_1 being totally ordered, $g^{\alpha}_{\beta} = g^{\alpha}_{\gamma} \kappa\, g^{\gamma}_{\beta}$ if $\beta \leq \gamma \leq \alpha$.

So in both cases we have constructed a functor G_{I_2} satisfying (i), (ii), (iii). This contradicts the maximality of (I_1, G_{I_1}), which proves that I_1 is I itself.

c) We can apply Theorem 2 to the functor $G = G_{I_1}$ itself, which gives us, for each α_0 in I, a unique g_{α_0} such that



commutes for $\alpha \geq \alpha_0$. To complete the proof, we'll show that g_α has the expected form, by induction on n. There is no problem if $n = 0$. Suppose that g_{α_0} satisfies the equality given in the theorem for B's of the form $B_{\alpha_0} \times \prod_{1\leq i\leq p} F_{\alpha_0+i} \times \prod_{\alpha>\alpha_0+p} E_\alpha$. Let

$$B' = B_{\alpha_0} \times \prod_{1\leq i\leq p+1} F_{\alpha_0+i} \times \prod_{\alpha>\alpha_0+p+1} E_\alpha .$$

Then

$$g_{\alpha_0}(\omega_{\alpha_0})(B') = g_{\alpha_0}^{\alpha_0+p+1}(\omega_{\alpha_0})(B_{\alpha_0} \times \prod_{1\leq i\leq p+1} F_{\alpha_0+i})$$

$$= g_{\alpha_0+p}^{\alpha_0+p+1} \kappa\, g_{\alpha_0}^{\alpha_0+p}(\omega_{\alpha_0})(B_{\alpha_0} \times \prod_{1\leq i\leq p+1} F_{\alpha_0+i})$$

$$= \int g_{\alpha_0+p}^{\alpha_0+p+1}(\cdot)(B_{\alpha_0} \times \prod_{1\leq i\leq p+1} F_{\alpha_0+i})\, d g_{\alpha_0}^{\alpha_0+p}(\omega_{\alpha_0}).$$

The induction hypothesis gives the form of $g_{\alpha_0}^{\alpha_0+p}(\omega_{\alpha_0})$ on the sets $B_{\alpha_0} \times \prod_{1\leq i\leq p} F_{\alpha_0+i}$.

From this, we can deduce that, for any characteristic function, hence for any simple function and finally for any bounded measurable function X from Ω_{α_0+p} to R :

$$\int X\, d g_{\alpha_0}^{\alpha_0+p} = \int d f_{\alpha_0}(\omega_{\alpha_0}) \ldots \int d f_{\alpha_0+p-1}(\omega_{\alpha_0}, x_{\alpha_0+1}, \ldots, x_{\alpha_0+p-1}) X.$$

If we apply this to $X = g_{\alpha_0+p}^{\alpha_0+p+1}(\cdot)(B_{\alpha_0} \times \prod_{i\leq p+1} F_{\alpha_0+i})$ defined by:

$$X(\omega_{\alpha_0+p}) = X(\omega_{\alpha_0}, x_{\alpha_0+1}, \ldots, x_{\alpha_0+p})$$

$$= \chi_{B_{\alpha_0}}(\omega_{\alpha_0}) \chi_{F_{\alpha_0+1}}(x_{\alpha_0+1}) \cdots \chi_{F_{\alpha_0+p}}(x_{\alpha_0+p}) f(\omega_{\alpha_0+p})(F_{\alpha_0+p+1})$$

we get

$$\chi_{B_{\alpha_0}}(\omega_{\alpha_0}) \int_{F_{\alpha_0+1}} d f_{\alpha_0}(\omega_{\alpha_0}) \ldots \int_{F_{\alpha_0+p+1}} d f_{\alpha_0+p}(\omega_{\alpha_0}, \ldots, x_{\alpha_0+p}).$$

This is what we expected and the proof is complete. Δ

b) *Remark.* If $I = \mathrm{N}$, the existence of the $g_{\alpha_0}(\omega_{\alpha_0})$ gives back Ionescu-Tulcea Theorem as stated in [4].

c) **Theorem 3 bis.** *Theorem 3 is still valid with $\mathcal{P}ol$ instead of $\mathfrak{M}es$ and $(\mathrm{N}, \geq)$ instead of $(I, \geq)$.*

Δ. 1° With the notations of Theorem 3, we first show that g_n^{n+1} is a morphism of $\mathcal{PT}$, that is, is continuous. It will follow from the more general result:

Proposition. Let Ω_1 and Ω_2 be objects of $\mathcal{P}ol$ and define

$$\theta : \Pi(\Omega_1) \times \Pi(\Omega_2) \to \Pi(\Omega_1 \times \Omega_2) : \quad (P_1, P_2) \mapsto P_1 \times P_2$$

(product probability measure). Then θ is continuous.

δ. The set

$$A = \{ f : \Omega_1 \times \Omega_2 \to \mathrm{R} \mid f(\omega_1, \omega_2) = \sum_{i=1}^{n} a_i f_{1i}(\omega_1) f_{2i}(\omega_2),\ f_{ij} : \Omega_i \to \mathrm{R} \text{ continuous bounded} \}$$

is an algebra containing the constant maps and separating the points of $\Omega_1 \times \Omega_2$; hence, by Stone-Weierstrass Theorem, it is dense in the set of continuous bounded maps from $\Omega_1 \times \Omega_2$ to R, endowed with the topology of uniform convergence on compact subsets.

Now let $((P_1^n, P_2^n))_n$ converge to (P_1, P_2), f be a continuous map from $\Omega_1 \times \Omega_2$ to R bounded by 1 and ϵ be a fixed positive real number. For $i = 1, 2$, the $(P_i^n)_n$ are uniformly tight; so there is a compact K_i such that $P_i^n(K_i) > 1 - \epsilon/16$ for all n ; if follows

$$(P_1^n \times P_2^n)(K_1 \times K_2) > 1 - \frac{\epsilon}{8} \quad \text{for each } n.$$

Choose a g in A such that

$$\sup_{K_1 \times K_2} | f(\omega_1, \omega_2) - g(\omega_1, \omega_2) | < \frac{\epsilon}{8},$$

bounded by 1. We have

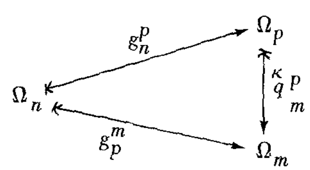
$$(3) \quad | \int f \, d(P_1^n \times P_2^n) - \int f \, d(P_1 \times P_2) | < \frac{3\epsilon}{4} + | \int g \, d(P_1^n \times P_2^n) - \int g \, d(P_1 \times P_2) | .$$

But since g has the form $\sum_{i=1}^{p} a_i g_{1i}(\omega_1) g_{2i}(\omega_2)$, from Fubini Theorem we get

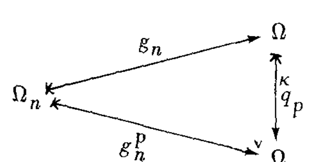
$$\int g \, d(Q_1 \times Q_2) = \sum_{i=1}^{p} a_i \int g_{1i} \, dQ_1 \int g_{2i} \, dQ_2 ,$$

so that $\int g \, d(P_1^n \times P_2^2)$ converges to $\int g \, d(P_1 \times P_2)$: it follows that there is an N such that the second member of (3) is less than ϵ for $n > N$. And θ is continuous. δ

2° So g_n^{n+1} is continuous. Now, if $p > n$, let us define the continuous map g_n^p as the composite $g_n^p = g_{p-1}^p \kappa \dots \kappa \, g_n^{n+1}$. The commutativity of all the diagrams



$(n \leq m \leq p)$ is straightforward (cf. Theorem 3). For every fixed n in N, Ω is the projective limit of the $(\Omega_p)_{p \geq n}$. It follows then, from Theorem 1 bis, that there exists a g_n such that



commutes for all $p \geq N$. The computation of g_{a_0} in Theorem 3 still applies here. Δ

III. RANDOM TOPOLOGICAL ACTIONS OF CATEGORIES.

In the above study, processes were always defined by a functor from an ordered set, representing time, to $\mathcal{PT}$. One could also imagine that between two times $t < t'$, several actions on the «moving point» were possible, in which case the process could be determined by a functor from a certain category C to $\mathcal{PT}$. This is the reason we now define the random topological actions, which are the «non-deterministic» analogues to the topological actions of a category on a topological space E. It is hoped they lead to applications in «non-deterministic» optimization problems similar to those studied by A. Ehresmann [2] in the «deterministic» case; this notion should as well be useful for a probabilistic generalization of stochastic automata.

Throughout this Part III, C is a category object in $\mathcal{P}ol$, that is, a category C endowed with a Polish topology for which

$$C_2 \xrightarrow{comp.} C_1 \underset{codom}{\overset{dom}{\rightrightarrows}} C_0$$

are continuous.

1. Definitions. Let E be an object of $\mathcal{P}ol$.

a) *A random topological action* (abbreviated in rta) (resp. a topological action (ta)) of C on E is a functor $\tilde{\ }$ from C to $\mathcal{PT}$ (resp. to $\mathcal{P}ol$) satisfying:

(i) There is a continuous map $p_\sim$ from E onto C_0 with $\tilde{e} = \bar{p}_\sim^{1}\{e\} = E_e$ for any $e \in C_0$.

(ii) If $C*E$ is defined by the pullback

$$\begin{array}{ccc} C*E & \longrightarrow & E \\ \downarrow & & \downarrow p_\sim \\ C & \xrightarrow{dom} & C_0 \end{array}$$

in $\mathcal{P}ol$ then, in $\mathcal{PT}$, the map $(f, x) \mapsto \tilde{f}(x)$ defines a morphism from $C*E$ to E.

b) *Remarks.* 1° A topological action is equivalent to an internal diagram in $\mathcal{P}ol$.

2° In the case of a rta, $\tilde{f}(x)$ is in fact in $\Pi(E_{codf})$, but this space is homeomorphic to, and identified with, the subspace of $\Pi(E)$ of those probability measures with support the fibre E_{codf}.

c) *Examples. 1°* If $\tilde{\ }$ is a ta, its composite with $\overset{\kappa}{\cdot}, \mathcal{P}ol \to \mathcal{PT}$ is a rta (of C on E).

2° A Markov process given by the morphisms $f_n : \Omega_n \leftarrow\!\!\!\rightarrow \Omega_{n+1}$ is a rta of the order $(\mathrm{N}, \geq)$ on the coproduct E of the state spaces Ω_n; indeed, $\tilde{\ }$ maps $(n, n+1)$ on f_n. More generally, if several actions were possible between time n and $n+1$, the category acting on E would still have N as set of its objects but there would be several morphisms between integers m and n, $m > n$ (cf. [2] for an example of such a category).

2. Topological action associated to a random topological action.

The domain of the internal diagram (or «catégorie d'hypermorphismes») associated to a ta on E is $C*E$ with composition

$$(f, x)(g, y) = (fg, y) \quad \text{iff} \quad x = \tilde{g}(y).$$

Looking for the corresponding notion in the case of a rta naturally leads to make C act on probability measures on E, which is possible thanks to the canonical functor:

$$\bar{\cdot}, \mathcal{PT} \to \mathcal{P}ol, (\Omega \xleftarrow{\ \theta\ } \Omega') \mapsto (\Pi(\Omega) \xrightarrow{\mu_{\Omega'} \circ \Pi(\theta)} \Pi(\Omega')).$$

a) In this section, we suppose given a rta $\tilde{\ }$ of C on E, and we denote by p the associated surjection p-: $E \to C_0$, by E' the subspace $\bigcup_{e \in C_0} \Pi(\bar{p}^1\{e\}) = \bigcup_{e \in C_0} \Pi(E_e)$ of $\Pi(E)$. This union being pairwise disjoint, one can define a map p' from E' onto C_0 by:

$$p'(P) = e \quad \text{iff} \quad P \in \Pi(E_e) \ (\text{iff} \ P(E_e) = 1).$$

b) **Proposition 1.** *E' is closed in $\Pi(E)$ (hence is polish) and p' is continuous from E' to C_0.*

Δ. Let $(P_n)_n$ be a sequence of E' converging to P in $\Pi(E)$ and $e_n = p'(P_n)$.

1° The set $\{P_n \mid n \in \mathrm{N}\} \cup \{P\}$ is compact, hence unifomly tight; in particular there is a compact K such that $P(K) > 1/2$ and $P_n(K) > 1/2$ for each n. If we choose an x_n in each $K \cap E_{e_n}$, the sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ which converges to an x in K; then $e = p(x) = \lim_n e_{n_k}$. If P_{n_k} is denoted by Q_k and e_{n_k} by e'_k, the sequences $(Q_k)_k$ and $(e'_k)_k$ respectively converge to P and e. We'll now prove that $P(E_e) = 1$ which will imply that $P \in \Pi(E_e) \subset E'$. If this was not true, there would be an $\epsilon > 0$ such that $P(E_e) < 1 - \epsilon$. E_e being closed in E metrizable, it has an open neighborhood U satisfying $P(\bar{U}) < 1 - \epsilon$. But $\bar{U}$ itself being a G_δ in E nomal, its characteristic map $\chi_{\bar{U}}$ is the pointwise decreasing limit of a sequence of continuous maps from E to $[0, 1]$. Hence there is such a map ϕ with value 1 on $\bar{U}$ such that $\int \phi \, dP < 1 - \epsilon$. But, since $(Q_k)_k$ converges to Q, there is an m in N such that $\int \phi \, dP_k < 1 - \epsilon$ for every $k > m$.

Let K' be a compact subspace of E satisfying $Q_k(K') > 1 - \epsilon$ for every k, and K_k its intersection with $E_{e'_k}$. If none of the sets K_k $(k > m)$ was contained in U, we could find an $y_k \in K_k$, $y_k \notin U$, for each k, and the sequence $(y_k)_k$ would have a subsequence converging to an y in K with $p(y) = e$. So y would be in E_e, and hence in U, which is absurd since $(y_k)_k$ is a sequence of the closed complement of U. Therefore there is a K_k $(k > m)$ contained in U; it follows that

$$1 - \epsilon < Q_k(K_k) \le Q_k(U) \le \int \phi \, dQ_k < 1 - \epsilon.$$

We have reached a contradiction, which means that $P(E_e) = 1$.

2° Suppose $e_n = p'(P_n)$ does not converge to $e = p'(P)$: there is a neighborhood V of e which contains no point of a subsequence $(e_{n_k})_k$ of $(e_n)_n$. p being continuous, there is, for each x in E_e, an open neighborhood V_x in E which intersects no $E_{e_{n_k}}$ (choose V_x such that $p(V_x) \subset V$). The union U of the V_x is an open neighborhood of E_e, so that there is a continuous map ϕ from E to $[0, 1]$ with value 1 on E_e and 0 on the complement of U. $U \cap E_{e_{n_k}}$ being empty, we have

$$\int \phi \, dP_{n_k} = 0 \qquad \text{and} \qquad \int \phi \, dP = 1,$$

which is absurd since $(P_{n_k})_k$ converges to P. Hence $p'(P_n)$ converges to $p'(P)$, and p' is continuous. Δ

c) **Theorem 1.** *If* $\hat{\ }$ *is a random topological action on* E, *its composite with the functor* $\hat{\ }: \mathcal{PT} \to \mathcal{P}_{\mathcal{A}}\ell$ *is a topological action* $\hat{\ }$ *on* E'.

Δ. Defining p' as in a, condition (i) of Definition 1 is satisfied from Proposition 1. It remains to show that $(f, P) \mapsto \hat{f}(P)$ is continuous from $C * E'$ (defined as the obvious pull-back) to $\Pi(E)$. It is enough to prove that, for each continuous map $\theta: E \to [0, 1]$, the map

$$T : C * E' \to [0, 1] : (f, P) \mapsto \int \theta \, d\hat{f}(P)$$

is continuous. Indeed, from Formula a of the proof of Theorem 1 (I-3):

$$\int \theta \, d\hat{f}(P) = \int_{E_e} \xi_\theta \circ \tilde{f} \, dP = \int_{E_e} [\int \theta \, d\tilde{f}(.)] \, dP,$$

with $e = dom(f)$. The map $(f, z) \mapsto \int \theta \, d\tilde{f}(z)$ being continuous from the closed subset $C * E$ of $C \times E$ to $[0, 1]$, it can be extended to a continuous map $\theta': C \times E \to [0, 1]$. Now

$$\Theta : C \times \Pi(E) \to [0, 1] : (f, P) \mapsto \int \theta'(f, .) \, dP$$

is an extension of T to $C \times \Pi(E)$. Hence it is enough to show it is continuous on $C * E'$: Let $((f_n, P_n))_n$ converge to (f, P) and ϵ be a fixed real positive number; for each n,

$$|\Theta(f_n, P_n) - \Theta(f, P)| \leq |\Theta(f_n, P_n) - \Theta(f, P_n)| + |\Theta(f, P_n) - \Theta(f, P)|.$$

Since $\Theta(f, .)$ is continuous (for $\theta'(f, .)$ is), there is an N such that

$$|\Theta(f, P_n) - \Theta(f, P)| < \frac{\epsilon}{2} \quad \text{for } n > N.$$

The set of P_n's being uniformly tight, there is a compact K in E such that

$$\inf_n P_n(K) > 1 - \frac{\epsilon}{8}.$$

Then, θ' being bounded by 1,

$$|\Theta(f_n, P_n) - \Theta(f, P_n)| \leq \int_K |\theta'(f_n, .) - \theta'(f, .)| \, dP_n + \frac{\epsilon}{4}.$$

θ' is uniformly continuous on the compact $K' = (\{f_n \mid n \in \mathbb{N}\} \cup \{f\}) \times K$, which implies that $\theta'(f_n, .)$ uniformly converges to $\theta'(f, .)$ on K; so there is N' such that

$$\sup_{z \in K} |\theta'(f_n, z) - \theta'(f, z)| < \frac{\epsilon}{4} \quad \text{for } n > N'.$$

Then

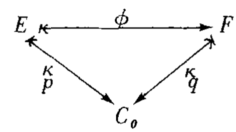
$$|\Theta(f_n, P_n) - \Theta(f, P)| < \epsilon \quad \text{for } n > sup(N, N').$$

It follows that Θ is continuous. Δ

d) *Remark.* The functor $\hat{\ }$ of Theorem 1 takes its values in the category $\mathcal{F}$ of free algebras of (Π, η, μ), so that not any ta on a set E' such that $E'_e = \Pi(E_e)$, with $(E_e)_{e \in C_0}$ a partition in closed sub-spaces of a Polish space E, actually comes from a rta. In fact, it can be shown that only those ta $\hat{\ }$ which factorize through $\mathcal{F}$ do, thanks to the isomorphism between $\mathcal{F}$ and $\mathcal{PT}$.

3. The category of random topological actions of C.

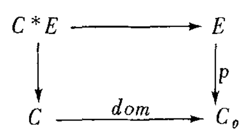
a) *Notations.* 1° The objects of the category $\mathcal{R}$ are the couples (E, p) where E is a Polish space and p a map from E *onto* C_0. A morphism $\phi : (E, p) \to (F, q)$ is a morphism $\phi : E \nleftarrow\!\!\to F$ such that



commutes. Composition is deduced from $\mathcal{PT}$.

2° $\mathcal{R}and$ has for objects the random topological actions of C. If $\dot{\cdot}$ is a rta on E and $\dot{\cdot}$ a rta on F, a morphism $\phi : \dot{\cdot} \to \dot{\cdot}$ is a morphism $\phi : (E, p\text{-}) \to (F, p\text{-})$ in $\mathcal{R}$ such that the family $(\phi_e)_{e \in C_0}$, where $\phi_e : E_e \leftarrow\!\!\!\to F_e$ is the restriction of ϕ to the fibers on e, considered with values in $\Pi(F_e)$, defines a natural transfomation $\dot{\cdot} \Rightarrow \dot{\cdot}$. This last condition means that ϕ commutes with the actions. Composition is again deduced from $\mathcal{PT}$.

b) **Theorem 2.** *The forgetful functor from $\mathcal{R}and$ to $\mathcal{R}$ has a left adjoint. The free object over (E, p) is the rta $\overset{\circ}{.}p$ on the topological subspace C^*E of $C \times E$ defined by the pullback*



given by $\overset{\circ}{g}p(f, x) = \eta_{C^*E}(gf, x)$ *iff* $dom\, g = codom\, f$.

Δ. Let us first prove that $\overset{\circ}{.}p$ (denoted here $\overset{\circ}{.}$) is a rta on C^*E. The map

$$p_0 : C^*E \to C_0 : (f, x) \to codom(f)$$

is onto and continuous, and for each e of C_0, we have $\overset{\circ}{e} = p_0^{-1}\{e\} = (C^*E)_e$. So condition (i) of Definition 1 is satisfied. For any $g : e \to e'$ in C, we can see $\overset{\circ}{g}$ as a morphism in $\mathcal{PT}$, from $(C^*E)_e$ to $(C^*E)_{e'}$. It is easy then to show that $\overset{\circ}{.} : C \to \mathcal{PT}$ is a functor. At last, the map

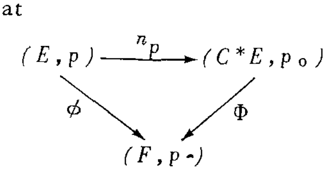
$$C^*(C^*E) \to C^*E : (g, (f, x)) \mapsto \overset{\circ}{g}(f, x)$$

is continuous since composition in C and η_{C^*E} are.

We now define

$$n_p : (E, p) \to (C^*E, p_0) : x \mapsto \eta_{C^*E}(p(x), x).$$

Let $\dot{\cdot}$ be a rta on F, and $\phi : (E, p) \to (F, p\text{-})$ any morphism if $\mathcal{R}$. If a morphism Φ from $\overset{\circ}{.}$ to $\dot{\cdot}$ in $\mathcal{R}and$ is such that



commutes, it satisfies necessarily $\Phi(p(x), x) = \phi(x)$, and, from the definition of $\mathcal{R}and$,

$$\Phi(f, x) = (\Phi \kappa \overset{\circ}{f})(p(x), x) = (\hat{f} \kappa \Phi)(p(x), x) = (\hat{f} \kappa \phi)(x),$$

for every (f, x) in C^*E. So Φ is unique if it exists. To show that the above defined Φ fits, it remains to check it is continuous. But since

$$\Phi(f, x) = [\mu_{F_{cod f}} \circ \Pi(\hat{f})](\phi(x)),$$

continuity of Φ follows from continuity of $(f,x) \mapsto (f, \phi(x))$ and of

$$(f,P) \mapsto \mu_{F_{cod f}} \circ \Pi(\hat{f})(P) = \hat{f}(P)$$

(cf. 2-c, Theorem 1). Δ

c) *Remark.* Let $\mathcal{R}_d$ be the subcategory of $\mathcal{R}$ with the same objects but with morphisms only the deteministic ones (of the fom $\overset{\circ}{\theta}$ with θ in $\mathcal{P}ol$), and $\mathcal{T}act$ the subcategory of $\mathcal{R}and$ with objects the ta (Example 1, c, 1°) and morphisms the deteministic maps between those. Then, by restriction, $\overset{o}{.}{}^{p}$ is still the free object over (E,p) for the forgetful functor from $\mathcal{T}act$ to $\mathcal{R}_d$; this is already known.

At last, we prove the following result, similar to the one obtained in the case of topological actions:

d) **Theorem 2.** *$\mathcal{R}and$ is (isomorphic to) the Eilenberg-Moore category of the monad generated by the above adjunction.*

Δ. Let us call this monad (P, n, m). For any $\phi: (F,q) \to (E,p)$, $P(\phi)$ is the only morphism in $\mathcal{R}and$ such that $n_p \kappa \phi = P(\phi) \kappa n_q$. Via the comparison functor from $\mathcal{R}and$ every rta $\bar{.}$ on E becomes an algebra: the structural arrow is given by

$$h(f,x) = \bar{f}(x) \quad \text{for any } (f,x) \in C^*E.$$

In particular, m_p is defined by

$$m_p(g,(f,x)) = \eta_{C^*E}(gf,x) \quad \text{for } (g,(f,x)) \in C^*(C^*E).$$

Every morphism in $\mathcal{R}and$ becomes a morphism of algebras as well. We now wanna prove the converse. Let us consider

$$\xi: (F,q) \to (E,p), \quad k: (C^*F, p_{o_q}) \to (F,q), \quad h: (C^*E, p_{o_p}) \to (E,p),$$

a morphism of algebras; we denote $k(f,y)$ by $\hat{f}(y)$ for $(f,y) \in C^*F$ and $h(f,x)$ by $\bar{f}(x)$ for $(f,x) \in C^*E$. Using the notations of Theorem 3 bis (II-4, c), we get the

Lemma. *For (f,y) in C^*F such that $q(y) = e$, we have:*

a) $(n_p \kappa \xi)(y) = \eta_C(e) \times \xi(y)$.

b) $P(\xi)(f,y) = \eta_C(f) \times \xi(y)$.

c) $(\xi \kappa k)(f,y) = (\xi \kappa \hat{f})(y)$.

d) $(h \kappa P(\xi))(f,y) = (\bar{f} \kappa \xi)(y)$.

δ. C^*E being a closed subset of $C \times E$, every element of $\Pi(C^*E)$ can be seen as an element of $\Pi(C \times E)$, hence is determined by its values on the subsets $B_C \times B_E$, where $B_C \in \mathcal{B}_C$ and $B_E \in \mathcal{B}_E$. For such a subset:

a) $(n_p \kappa \xi)(y)(B_C \times B_E) = \int n_p(.)(B_C \times B_E)\, d\xi(y)$

$= \int \eta_{C^*E}(e,.)(B_C \times B_E)\, d\xi(y)$ (for $\xi(y)$ is concentrated on E_e)

$= \eta_C(e)(B_C). \xi(y)(B_E)$.

b) $P(\xi)(f,y)(B_C \times B_E) = \int \overset{o}{f}{}^{p}(.,.)(B_C \times B_E)\, d(n_p \kappa \xi)(y)$

$$= \int \overset{\circ}{f}^{p}(e,.)(B_C \times B_E)\, d\xi(y) \quad \text{from a and Fubini Theorem}$$

$$= \int \eta_{C*E}(f,.)(B_C \times B_E)\, d\xi(y) = \eta_C(f)(B_C).\xi(y)(B_E).$$

c) $(\xi \kappa k)(f,y) = (\mu_E \circ \Pi(\xi))(k(f,y)) = (\mu_E \circ \Pi(\xi))(\hat{f}(y)) = (\xi \kappa \hat{f})(y).$

d) $(h \kappa \mathrm{P}(\xi))(f,y)(B_E) = \int h(.,.)(B_E)\, d\mathrm{P}(\xi)(f,y) = \int h(f,.)(B_E)\, d\xi(y)$

$$= \int \bar{f}(.)(B_E)\, d\xi(y) = (\bar{f} \kappa \xi)(y)(B_E). \quad \delta$$

Applying c and d of this Lemma to an algebra $((E,p),h)$ with

$$(F,q) = (C^*E, p \circ_p), \quad \xi = h, \quad k = m_p,$$

we get

$$(h \kappa m_p)(f,(g,x)) = (h \kappa \overset{\circ}{f}^{p})(g,x) = h(fg,x) = \widetilde{fg}(x)$$

and

$$(h \kappa \mathrm{P}(h))(f,(g,x)) = (\bar{f} \kappa h)(g,x) = (\mu_E \circ \Pi(\bar{f}))(\bar{g}(x)) = (\bar{f} \kappa \bar{g})(x),$$

so that $\bar{\ }$ is a functor to $\mathcal{PT}$ (the fact that it sends units to units follows from the equality $h \kappa n_p = \eta_E$), and hence a rta on E.

To prove that any morphism of algebras comes from a morphism in $\mathfrak{R}and$, we use c and d of the Lemma again, applied this time to such a morphism ξ between algebras $((F,q),k)$ and $((E,p),h)$, considered as rta $\bar{\ }$ and $\bar{\ }$ respectively. From c and d, ξ commutes with the actions and so is a morphism in $\mathfrak{R}and$. Δ

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