Ionization in a 1-Dimensional Dipole Model

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Abstract

We study the evolution of a one dimensional model atom with δ-function binding potential, subjected to a dipole radiation field $E(t)x$ with $E(t)$ a $2\pi/\omega$-periodic real-valued function. We prove that when $E(t)$ is a trigonometric polynomial, complete ionization occurs, i.e. the probability of finding the electron in any fixed region goes to zero as $t \to \infty$.

For $\psi(x, t = 0)$ compactly supported and general periodic fields, we decompose $\psi(x, t)$ into uniquely defined resonance terms and a remainder. Each resonance is $2\pi/\omega$ periodic in time and behaves like the exponentially growing Green's function near $x = \pm\infty$. The remainder is given by an asymptotic power series in $t^{-1/2}$ with coefficients varying with x.

1 Introduction

The ionization of an atom by an electromagnetic field is one of the central problems of atomic physics. There are a variety of approximate methods for treating this problem, including perturbation theory (Fermi's golden rule), numerical integration of the time-dependent Schrödinger equation and semi-classical phase space analysis leading to stochastic ionization [3, 5, 18, 21, 22, 25]. Rigorous approaches include Floquet theory and complex dilations [18, 19, 37, 2]. Despite this, there are few exact results available for the ionization of a bound particle by a realistic time-periodic electric field of dipole form $\vec{E}(t) \cdot \vec{x}$ (an AC-Stark field) for fields of arbitrary strength. The most realistic results we are aware of are based on complex scaling [18, 19, 37] and show ionization (for small electric field) of certain bound states of the Coulomb atom as well as defining resonances in some regions of the complex energy plane.

The lack of rigorous results for large electric fields is true not only for realistic systems with Coulombic binding potential, but even for model systems with short range binding potentials [3, 5, 17]. The most idealized version of the latter has an attractive δ-function potential in 1 dimension. The unperturbed Hamiltonian $H_0 = -\partial_x^2 - 2\delta(x)$ has a bound state $\phi_0(x) = e^{-|x|}$ with energy $E_0 = -1$, and explicitly known continuum states [9]. This model has been studied extensively in the literature, but the only rigorous results (known to us) concerning ionization involve short range external forcing potentials rather than dipole interaction; see however [4, 15, 24] for some rigorous bounds on the ionization probability by a dipole potential for finite time pulses. Detailed results for compactly supported forcings were obtained in [8, 9, 28, 6]. In this work we develop techniques to deal with physically realistic dipole interactions.

We consider the time evolution of a particle in one dimension governed by the Schrödinger equation (in appropriate units) with a time-periodic dipole field:

$$i\partial_t\psi(x,t) = \left(-\frac{\partial^2}{\partial x^2} - 2\delta(x)\right)\psi(x,t) + E(t)x\psi(x,t) \tag{1.1a}$$

$$\psi(x,0) = \psi_0(x) \in L^2(\mathbb{R}) \tag{1.1b}$$

We prove the following result.

Theorem 1 (Ionization) *Suppose $E(t)$ is a trigonometric polynomial, i.e.*

$$E(t) = \sum_{n=1}^{N}\left(E_n e^{in\omega t} + \overline{E_n}e^{-in\omega t}\right). \tag{1.2}$$

Then for any $\psi_0(x) \in L^2(\mathbb{R})$ ionization occurs, i.e. for $\psi(x,t)$ solving (1.1),

$$\lim_{t\to\infty}\int_{-L}^{L}|\psi(x,t)|^2 dx = 0, \qquad \forall L \in \mathbb{R}^+ \tag{1.3}$$

If $\psi_0(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the approach to zero is at least as fast as t^{-1}.

When $E(t)$ is not a trigonometric polynomial (i.e. $N = \infty$ in (1.2)), the Floquet Hamiltonian may have time-dependent bound states and ionization may fail. This is uncommon, but there are examples of time periodic operators where such bound states exist [9, 27].

A key part of the proof of Theorem 1 is a theorem characterizing the structure of $\psi(x,t)$. This result holds even if $N = \infty$ in (1.2). We first need a definition.

Definition 1.1 *For the system described by* (1.1)*, a Gamow vector is a classical solution of Floquet eigenvalue equation with $\Re\check{\sigma}_k \in [0,\omega)$*[1]*:*

$$\left(-i\partial_t - \frac{\partial^2}{\partial x^2} - 2\delta(x) + E(t)x - \check{\sigma}_k\right)\Phi_{k,0}(x,t) = 0 \tag{1.4a}$$

$$\Phi_{k,0}(x,t) = \Phi_{k,0}(x, t + 2\pi/\omega), \tag{1.4b}$$

subject to a radiation boundary condition at $x = \pm\infty$:

$$\Phi_{k,0}(x,t) = \begin{cases} \sum_n \psi_n^L e^{i\sqrt{\check{\sigma}_k + n\omega}x} e^{-in\omega t} e^{-ib(t)x - ia(t)} & x \le 0 \\ \sum_n \psi_n^R e^{-i\sqrt{\check{\sigma}_k + n\omega}x} e^{-in\omega t} e^{-ib(t)x - ia(t)} & x \ge 0 \end{cases} \tag{1.4c}$$

[1] We make this restriction due to the time-modulation invariance of (1.4). If $\Phi_{k,0}(x,t)$ solves (1.4), then $e^{in\omega t}\Phi_{k,0}(x,t)$ also solves (1.4) if we replace $\check{\sigma}_k$ by $\check{\sigma}_k + n\omega$.

The functions $a(t)$ and $b(t)$ are defined in (1.8), and are necessary to account for the presence of the $E(t)x$ potential. The square root is chosen to have a branch cut on $-i\mathbb{R}^+$ and maps $\mathbb{R}^+$ into $\mathbb{R}^+$.

A generalized Gamow vector $\Phi_{k,j}(x,t)$ solves the equation:

$$\left(-i\partial_t - \frac{\partial^2}{\partial x^2} - 2\delta(x) + E(t)x - \check{\sigma}_k\right)\Phi_{k,j}(x,t) = \Phi_{k,j-1}(x,t) \tag{1.5}$$

By definition, $\Phi_{k,-1}(x,t) = 0$.

Remark 1.2 For $\Im\check{\sigma}_k \geq 0$, (1.4c) implies that solutions to (1.4) revert to ordinary $L^2([0, 2\pi/\omega] \times \mathbb{R})$-eigenvalues of the Floquet Hamiltonian. This implies that solutions of (1.4) with $\Im\check{\sigma}_k > 0$ are impossible; if such solutions existed, that would contradict the L^2-self-adjointness of (1.4a). When $\Im\check{\sigma}_k = 0$, Gamow vectors revert to being $L^2([0, 2\pi/\omega] \times \mathbb{R})$ solutions of (1.4a), and $\psi_n^{R,L} = 0$ for $n \geq 0$ (otherwise $\Phi_{k,0}(x,t)$ would not be in L^2). When $\Im\check{\sigma}_k < 0$ the terms in (1.4c) become exponentially growing near $x = \pm\infty$, and no exponentially decaying terms are present. This is what distinguishes Gamow vectors from other solutions to (1.4a) and (1.4b).

In fact, one can construct exponentially growing solutions of (1.4a) for *all* σ. Such solutions would have both growing and decaying terms on at least one side of zero, and therefore fail to satisfy (1.4c). This makes the radiation boundary condition (1.4c) crucial in defining Gamow vectors and resonances [25, 31, 20, 32, 36]. These boundary conditions define $\check{\sigma}_k$ and $\Phi_{k,j}(x,t)$ uniquely. Theorem 2 (below) shows that these Gamow vectors are directly connected to the time behavior of $\psi(x,t)$.

Theorem 2 *Suppose $\psi_0(x)$ is compactly supported and in H^1 and $E(t)$ is smooth and time periodic. Then, the solution $\psi(x,t)$ of (1.1), can be uniquely (for fixed M) decomposed as:*

$$\psi(x,t) = \sum_{k=0}^{M}\sum_{j=0}^{n_k} \alpha_{k,j} t^j e^{-i\check{\sigma}_k t}\Phi_{k,j}(x,t) + \Psi_M(x,t) \tag{1.6}$$

The resonance energies $\check{\sigma}_k$ satisfy $\Im\check{\sigma}_k \leq 0$, and are ordered according to $\Im\check{\sigma}_{k+1} \leq \Im\check{\sigma}_k$. The resonant states $\Phi_{k,j}(x,t)$ are the Gamow vectors (generalized, if $j > 0$), as per Definition 1.1.

In (1.6), we collect the M resonances with $\Im\check{\sigma}_k$ closest to zero, and M must be large enough so that we collect all resonances $\check{\sigma}_k$ with $\Im\check{\sigma}_k = 0$. The number of Gamow vectors (and resonances) may be infinite, but we can only include finitely many of them in (1.6).

The remainder $\Psi_M(x,t)$ has the following asymptotic expansion in time:

$$\Psi_M(x,t) \sim \sum_{j\in\mathbb{Z}} e^{ij\omega t}\sum_{n=1}^{\infty} D_{j,n}(x)t^{-n/2} \tag{1.7}$$

This expansion is uniform on compact sets in x, but not in L^2. In general, $D_{j,n}(x)$ is not in L^2.

Uniqueness of the decomposition is defined relative to the analytic structure of $\psi(x,t)$: the Zak transform of $\psi(x,t)$ has poles at $\sigma = \check{\sigma}_k$ (with residues proportional to $\Phi_{k,j}(x,t)$), while $\Psi_M(x,t)$ has Zak transform that is analytic on the region $\{\sigma : \Re\sigma \in (0,\omega), \Im\sigma > \Im\check{\sigma}_M\}$. The Zak transform is defined in Section 3.

Gamow vectors have a long history in quantum mechanics, dating back to [16, 13, 25]. They were first introduced by Gamow, who used them to study tunneling rates. Definition 1.1 is an extension of the usual definition of Gamow vectors; a Gamow vector for a compactly supported potential (on $[-L, L]$) is typically defined as a classical solution of the equation $[-\partial_x^2 + V(x) - \mu]\Phi(x) = 0$ having the behavior $\Phi(x) = \psi^L e^{i\sqrt{\mu}x}$ for $x < -L$ and $\Phi(x) = \psi^R e^{-i\sqrt{\mu}x}$ for $x > L$ (with μ the corresponding complex eigenvalue). The usual interpretation (see [32, 25] for an operator theoretic perspective) states that the real part of μ is the energy of a Gamow vector, while the imaginary part is the decay rate. In spite of their age and usefulness in making experimental predictions, rigorous justification of the use of Gamow vectors is still lacking (see, however [32]). Theorem 2 provides a rigorous definition of resonances and ionization rates for the case we consider, and is thus a step towards making Gamow vectors completely rigorous.

The remainder term $\Psi_M(x,t)$, which we shall term the dispersive part, incorporates any resonances $\check{\sigma}_k$ with $k > M$ (if such resonances exist), as well as the integral around a branch point which gives rise to the polynomially decaying component. The resonant states $\Phi_{k,j}(x,t)$ are the residues of the poles of that same function (M can not be greater than the total number of poles). This is equivalent (via (3.7)) to the requirement that the Laplace transform of $\Psi_M(x,t)$ is analytic for $\Im p > -\gamma_M$ except for an array of branch points on the periodic array $p = n\omega$, $n \in \mathbb{Z}$ (with p dual to t).

We believe the dispersive part $\Psi_M(x,t)$ is Borel summable, although this does not follow from our results. To show this, one needs to find exponential bounds on $\mathcal{Z}[\psi(0,t)](\sigma,t)$ as $\Im\sigma \to -\infty$, which would also show that there is only one resonance $\check{\sigma}_0$, the analytic continuation of the bound state.

Remark 1.3 Define $\gamma_k = -\Im\check{\sigma}_k$; for small values of γ_k, $2\gamma_k$ gives the dominant part of the ionization rate for the k-th resonance. The smallest rate, γ_0, gives the overall ionization rate for most experimentally relevant times [3]. We are only aware of one experiment where the dispersive part of the wavefunction has actually been observed experimentally [29], although under significantly different physical conditions[2]. In most experiments the dispersive part is small enough to be safely neglected, and is in fact very difficult to measure.

Remark 1.4 If $\psi_0(x)$ is not compactly supported a similar decomposition to (1.6) can be computed, but with extra terms coming from singularities of the

[2]In [29], the authors studied luminescent decay of dissolved organic materials after a pulsed laser excitation, and observed polynomial decay after all exponential terms had vanished.

Fourier transform of $e^{i\partial_x^2 t}\psi_0(x)$ (with respect to t). These terms are present even in the absence of a potential, and are therefore not resonances.

1.1 Small Field Limit

Replacing $E(t)$ by $\epsilon E(t)$, $\check{\sigma}_0$ and $\Phi_{0,0}(x,t)$ have convergent power series expansions in ϵ when $\omega^{-1} \notin \mathbb{N}$. When $\epsilon \to 0$, we have $e^{-i\check{\sigma}_0 t}\Phi_{0,0}(x,t) \to e^{it}e^{-|x|}$ (pointwise), the bound state of H_0 and $\Psi_1(x,t)$ goes to the projection of $\psi(x,t)$ on the continuum states of H_0. This shows that the first resonance is the analytic continuation in ϵ of the bound state. This rigorously justifies some standard physics calculations in [13, 16, 25] (see also the forthcoming work [36], from which we drew inspiration). The Fermi golden rule and multiphoton generalizations can be recovered in our formalism through perturbation theory.

All other resonances come from $\sigma = -i\infty$, i.e. as $\epsilon \to 0$, $\gamma_k \to \infty$ for $k \geq 1$. We conjecture that states with $k > 0$ (which do not exist when $\epsilon = 0$) do not exist regardless of ϵ. Indeed, in all other cases considered [8, 9], such states do not exist, but our technique does not rule them out. See Remark 3.15 for more details on this point.

1.2 Equivalent formulations

Here we describe some equivalent formulations of (1.1). This material is essentially taken from chapter 7 of [11]. We will use (1.10) in the proof of Theorem 1 and (1.9) in the proof of Theorem 2. We first define some auxiliary functions:

$$a(t) = \int_0^t b(s)^2 ds \equiv a_0 t + a_v(t) \tag{1.8a}$$

$$b(t) = \sum_{n=1}^{\infty} \left(\frac{E_n}{in\omega} e^{in\omega t} + \frac{\bar{E}_n}{-in\omega} e^{-in\omega t} \right) \tag{1.8b}$$

$$c(t) = 2\sum_{n=1}^{\infty} \left(\frac{E_n}{(in\omega)^2} e^{in\omega t} + \frac{\bar{E}_n}{(-in\omega)^2} e^{-in\omega t} \right) \equiv \sum_{n=1}^{\infty} \left(C_n e^{in\omega t} + \bar{C}_n e^{-in\omega t} \right) \tag{1.8c}$$

where $a_v(t)$ is $2\pi/\omega$ periodic and has mean 0, and $a_0 = (\omega/2\pi)\int_0^{2\pi/\omega} b(s)^2 ds$. Note that $(1/2)c''(t) = b'(t) = E(t)$.

Define $\psi_v(x,t) \equiv e^{+ia(t)}e^{+ib(t)(x-c(t))}\psi(x-c(t),t)$; then the following equation for ψ_v is equivalent to (1.1):

$$i\partial_t \psi_v(x,t) = \left(-\frac{\partial^2}{\partial x^2} - 2\delta(x - c(t)) \right) \psi_v(x,t) \tag{1.9}$$

This is the velocity gauge, and the equivalence can be verified by a computation[3]. Similarly, there is an equivalent equation in the magnetic gauge. We obtain it

[3]Equation (1.9) differs from what one finds in [11]. In [11], the authors take $\tilde{b}(t) = \int_0^t E(s)ds$ and $\tilde{c}(t) = \int_0^t b(t)dt$, which imply that $\tilde{c}(t) = c(t) + c_0 + c_v t$. This does not change the essential feature that $(1/2)c''(t) = b'(t) = E(t)$.

by setting $\psi_B(x,t) = e^{+ia(t)}e^{+ib(t)x}\psi(x,t)$:

$$i\partial_t \psi_B(x,t) = \left(-\frac{\partial^2}{\partial x^2} - 2\delta(x) + 2ib(t)\partial_x\right)\psi_B(x,t) \tag{1.10}$$

Remark 1.5 Suppose that either $\psi_B(x,t)$ or $\psi_v(x,t)$ are time-periodic solutions of (1.10) or (1.9). Then $\psi(x,t)$ is a time quasi-periodic solution of (1.1), and $e^{ia_0 t}\psi(x,t)$ is time-periodic.

These computations are formal, and we must show that at least one of (1.1), (1.9) or (1.10) are well posed. This is shown in Appendix C. Once one is well posed, all are, simply by applying the unitary gauge transformations.

1.3 Organization of the Paper

The paper is organized in the following way. In Section 2, we assume Theorem 2 to be true and use it to prove Theorem 1. In Section 3 we prove Theorem 2. In Section 4 we make some concluding remarks, and discuss possible directions of future research. Some technical material is presented in the appendices.

2 Ionization

Based on Theorem 2, we will show that the Floquet equation (1.4) in the magnetic gauge has no nonzero solutions with $\Im\sigma = 0$ which decay at $x = \pm\infty$. This implies ionization for compactly supported initial data. Ionization follows for all initial data in $L^2(\mathbb{R})$ by a simple application of the following well known result to the operator family $T(t) = 1_{[-L,L]}(x)U(t)$ (with $U(t)$ the propagator for (1.1)):

Proposition 2.1 *If $T(t)$ is a uniformly bounded family of operators on $L^2(\mathbb{R})$, and if $T(t)u \to 0$ for u in a dense subset of $L^2(\mathbb{R})$, then $T(t)u \to 0$ for all $u \in L^2(\mathbb{R})$.*

In Section 2.1, we solve (1.10) without a binding potential (the $-2\delta(x)$ term) and characterize the solutions. We then assume that a bound state $\Phi_{k,0}(x,t)$ exists, expand it in an appropriate basis, and derive necessary conditions on the coefficients to meet the boundary conditions (decay at $x = \pm\infty$ and continuity at $x = 0$).

In Section 2.2, we use the characterization of solutions we constructed in Section 2.1 and show for $E(t)$ a trigonometric polynomial that there are no continuous, nonzero solutions to (1.10) which vanish at $x = \pm\infty$. The basic technique is to analytically continue, in the t variable, both $\psi_B(0_-,t)$ and $\psi_B(0_+,t)$ (which must coincide) and use the Phragmen-Lindelöf theorem to show that an associated function must be entire and bounded (and therefore constant). This implies that any localized solution to (1.4) is zero, and ionization occurs.

2.1 Solutions to the free problem

By Theorem 2, we need to show that (1.4) has no nontrivial solutions. In the magnetic gauge, this is the same as showing that if $\Phi_{k,0}(x,t)$ solves

$$\check{\sigma}_k \Phi_{k,0}(x,t) = (-i\partial_t - \partial_x^2 - 2\delta(x) + 2ib(t)\partial_x)\Phi_{k,0}(x,t), \tag{2.1}$$

is time periodic and decays at $x = \pm\infty$, then $\Phi_{k,0}(x,t) = 0$.

We begin by solving (2.1) without the δ-function binding potential (and letting $\sigma = \check{\sigma}_k$, which causes no confusion in this section),

$$\sigma\psi(x,t) = (-i\partial_t - \partial_x^2 + 2ib(t)\partial_x)\psi(x,t) \tag{2.2}$$

Taking $\psi(x,t) = e^{\lambda x}\varphi_\lambda(t)$ as an ansatz, we obtain an ODE for $\varphi_\lambda(t)$:

$$\partial_t \varphi_\lambda(t) = -i\left(-\sigma - \lambda^2 + 2i\lambda b(t)\right)\varphi_\lambda(t) \tag{2.3}$$

This has the following family of solutions (recalling that $c'(t) = 2b(t)$):

$$\begin{aligned} \varphi_\lambda(t) &= e^{-iE_\lambda t}e^{\lambda c(t)} \\ E_\lambda &= -\sigma - \lambda^2 \end{aligned} \tag{2.4}$$

To ensure $2\pi/\omega$ periodicity in time, we must have $(-\sigma - \lambda^2) = m\omega$, $m \in \mathbb{Z}$. This implies that $\lambda = \pm i\sqrt{m\omega + \sigma}$ (with the branch cut of $\sqrt{z}$ taken to be $-i\mathbb{R}^+$). Therefore, (2.2) has the family of solutions:

$$\varphi_{m,\pm}(x,t) = e^{\pm\lambda_m x}e^{-im\omega t}e^{\pm\lambda_m c(t)} \tag{2.5a}$$

$$\lambda_m = -i\sqrt{\sigma + m\omega} \tag{2.5b}$$

2.2 Matching solutions

Given the family of solutions to (2.2), we can attempt to solve (1.10). Applying Theorem 2, we have three boundary conditions to satisfy:

$$\Phi_{k,0}(0,t) = \Phi_{k,0}(0_-,t) = \Phi_{k,0}(0_+,t) \tag{2.6a}$$

$$\partial_x \Phi_{k,0}(0_+,t) - \partial_x \Phi_{k,0}(0_-,t) = -2\Phi_{k,0}(0,t) \tag{2.6b}$$

$$\lim_{x\to\infty} \Phi_{k,0}(-x,t) = \lim_{x\to\infty} \Phi_{k,0}(+x,t) = 0 \tag{2.6c}$$

Consider now a solution $\Phi_{k,0}(x,t)$. We can expand (formally) $\Phi_{k,0}(x,t)$ in terms of the functions $\varphi_{m,\pm}$ in the regions $x < 0$ and $x > 0$ separately[4]:

$$\Phi_{k,0}(x,t) = \begin{cases} \sum_{m\in\mathbb{Z}}(\psi^L_{m,+}\varphi_{m,+}(x,t) + \psi^L_{m,-}\varphi_{m,-}(x,t)), & x \le 0 \\ \sum_{m\in\mathbb{Z}}(\psi^R_{m,+}\varphi_{m,+}(x,t) + \psi^R_{m,-}\varphi_{m,-}(x,t)), & x \ge 0 \end{cases} \tag{2.7}$$

For $m \ge 0$ (recalling $\check{\sigma}_k \in [0,\omega)$ and examining (2.5b)), the functions $\varphi_{m,\pm}(x,t)$ are oscillatory in x as $x \to \pm\infty$. Thus, if the coefficients $\psi^{L,R}_{m,\pm}$

[4] The validity of the expansion is proved in Lemma B.2 in Appendix B.

($m \geq 0$) were not zero, then $\Phi_{k,0}(x,t)$ would not decay as $x \to \pm\infty$, violating (2.6c).

Similarly, we observe that $\varphi_{m,+}(x,t)$ are exponentially growing when $m < 0$ as $x \to +\infty$, so $\psi^R_{m,+}$ must similarly be zero. The same argument applied to the region $x < 0$ shows that $\psi^L_{m,+}$ must be zero when $m < 0$. Therefore after dropping the $\pm$ in the coefficients $\psi^{L,R}_{m,\pm}$, we obtain the result we seek.

Thus, we find that we can actually write $\Phi_{k,0}(x,t)$ as:

$$\Phi_{k,0}(x,t) = \begin{cases} \sum_{m<0} \psi^L_m \varphi_{m,+}(x,t), & x \leq 0 \\ \sum_{m<0} \psi^R_m \varphi_{m,-}(x,t), & x \geq 0 \end{cases} \tag{2.8}$$

with both sequences $\psi^{L,R}_m$ in l^2. Although this derivation is purely formal, it is proved in Appendix B. It also motivates (1.4c).

Substituting (2.8) into the continuity condition (2.6a) yields:

$$\sum_{m<0} \psi^L_m e^{-im\omega t} e^{\lambda_m c(t)} = \sum_{m<0} \psi^R_m e^{-im\omega t} e^{-\lambda_m c(t)} \tag{2.9}$$

Proposition 2.2 *Suppose $E(t)$ is a trigonometric polynomial with highest mode N, that is $E(t) = \sum_{n=1}^{N}(E_n e^{in\omega t} + \bar{E}_n e^{-in\omega t})$. Set $z = e^{-i\omega t}$. Then $\Phi_{k,0}(0,t)$ has the decomposition:*

$$\Phi_{k,0}(0,t) = f(z) + g(z^{-1}) \tag{2.10}$$

The functions $f(\cdot)$ and $g(\cdot)$ are entire functions of exponential order $2N$, and $g(0) = 0$. This shows in particular that $\Phi_{k,0}(0,t)$ is continuous.

The correspondence between $\Phi_{k,0}(0,t)$, $f(z)$ and $g(z)$ is as follows. Let ψ_j denote the j'th Fourier coefficient of $\Phi_{k,0}(0,t)$, that is $\Phi_{k,0}(0,t) = \sum_j \psi_j e^{ij\omega t}$. Then letting f_j, g_j be the Taylor coefficients of $f(z)$, $g(z)$, we find $f_j = \psi_{-j}$ for $j \geq 0$ and $g_j = \psi_j$ for $j < 0$.

The proof of this fact uses results from Section 3, and is deferred to Appendix A. Finally, we state a result we use, proved in most complex analysis textbooks, e.g. [35].

Theorem 3 (Phragmen-Lindelöf) *Let $f(z)$ be an analytic function of exponential order $2N$, that is $|f(z)| \leq Ce^{C'|z|^{2N}}$. Let S be a sector of opening smaller than $\pi/2N$. Then:*

$$\sup_{z \in \partial S} |f(z)| \geq \sup_{z \in S} |f(z)|$$

We are now prepared to prove the main result.

Proof of Theorem 1.

We describe first the case $N = 1$ now (i.e. $E(t) = E\cos(\omega t)$; the case of arbitrary N is treated below). The key idea is that we can use (2.8) and (2.9) to obtain an asymptotic expansion of $\Phi_{k,0}(0_+,t)$ and $\Phi_{k,0}(0_-,t)$ in the open right and left half planes in the variable $z = e^{-i\omega t}$. To leading order as $|z| \to$

∞ in the left and right half planes (respectively), $\Phi_{k,0}(0_-,t) \sim \psi_m^L z^m e^{-C|\Re z|}$ and $\Phi_{k,0}(0_+,t) \sim \psi_m^R z^m e^{-C|\Re z|}$ (note that m and C may be different). This asymptotic expansion shows that $f(z)$ decays exponentially along any ray $z = re^{i\phi}$ in the open left or right half planes.

In fact, the asymptotic expansion allows us to observe that $f(z)$ (the part of $\Phi_{k,0}(0,t)$ which is analytic in z) must be bounded except possibly on the line $i\mathbb{R}$. Theorem 3 combined with Proposition 2.2 allow us to conclude that $f(z)$ is bounded on the line $i\mathbb{R}$. This shows $f(z)$ is bounded on $\mathbb{C}$ and hence zero.

Since $f(z)$ is zero, $\Phi_{k,0}(0,t) = g(z) \sim g_M z^{-M}$ for some $M \in \mathbb{N}$ (since $g(z)$ is analytic). But we previously showed also that $\Phi_{k,0}(0,t) \sim \psi_m^L z^m e^{-C|\Re z|}$. Two asymptotic expansions must agree to leading order; the only way this can happen is if $g(z) = \Phi_{k,0}(0,t) = 0$.

The main difference between the case $N = 1$ (monochromatic field) and $N > 1$ (polychromatic field) is that instead of the exponential asymptotic expansions being valid in the left and right half planes, they are valid in sectors of opening π/N; to show this we need to apply Theorem 3 to the boundaries of these sectors.

We now go through the details.

Step 1: Setup

Let $\Phi_{k,0}(x,t)$ be a solution to (1.4). By the hypothesis of Theorem 1, we let $E(t)$ be a nonzero trigonometric polynomial of order N. Let $z = e^{-i\omega t}$. Let $\mathfrak{C}(z) = \sum_{j=1}^{N} \left(\bar{C}_j z^j + C_j z^{-j} \right)$ where the C_j are the coefficients from (1.8c). We apply Proposition 2.2 to $\Phi_{k,0}(0,t)$ and (2.9) to obtain:

$$\begin{aligned} \Phi_{k,0}(0,t) &= f(z) + g(z^{-1}) \\ &= \sum_{m<0} \psi_m^L z^m e^{+\lambda_m \mathfrak{C}(z)} = \sum_{m<0} \psi_m^R z^m e^{-\lambda_m \mathfrak{C}(z)} \end{aligned} \tag{2.11}$$

The first equality holds by (2.10), the second by (2.8) with $x = 0$. A priori, equality holds only when $|z| = 1$. However, both of the latter two sums are analytic in any neighborhood of the unit circle in which they are uniformly convergent. Thus, $f(z) + g(z^{-1})$ is the analytic continuation of the sum if the sum is convergent in some neighborhood containing part of the unit disk.

For the rest of this proof, we make the following convention. The functions $\psi^{L,R}(z)$ are defined by

$$\psi^L(z) = \sum_{m<0} \psi_m^L z^m e^{+\lambda_m \mathfrak{C}(z)} \tag{2.12a}$$

$$\psi^R(z) = \sum_{m<0} \psi_m^R z^m e^{-\lambda_m \mathfrak{C}(z)} \tag{2.12b}$$

for those z for which the sum is convergent.

Step 2: Convergence of the sum

We show now that the sum in (2.11) is convergent in a sufficiently large region.

For $|z| \geq 1$ and $\Re\mathfrak{C}(z) > 0$, consider the sum $\sum_{m<0} \psi^R_m z^m e^{-\lambda_m \mathfrak{C}(z)}$. In this region, since $\Re\mathfrak{C}(z) > 0$, we find that $e^{\lambda_m \mathfrak{C}(z)} \leq 1$. The coefficients $\psi^{L,R}_m$ are bounded uniformly in m (since they form an l^2 sequence). For $|z| > 1$, z^m is geometrically decaying as $m \to -\infty$. Therefore the series is absolutely convergent when $|z| > 1$ and $\Re\mathfrak{C}(z) > 0$.

The same statement holds with $\sum_{m<0} \psi^L_m z^m e^{+\lambda_m \mathfrak{C}(z)}$ in the region where $\Re\mathfrak{C}(z) < 0$.

Let us define the following sets:

$$S^+ = \text{Connected component of } S^1 \text{ in } \{z \in \mathbb{C} : |z| \geq 1, \Re\mathfrak{C}(z) > 0\}$$

$$S^- = \text{Connected component of } S^1 \text{ in } \{z \in \mathbb{C} : |z| \geq 1, \Re\mathfrak{C}(z) < 0\}$$

A plot indicating the structure of these sectors (for a particular choice of $\mathfrak{C}(z)$) is shown in Figure 1 for the case where $N = 2$.

By Proposition 2.2, we see that $\psi^R(z)$ is analytic in S^+ and $\psi^L(z)$ is analytic in S^-, since the sum in (2.12) is convergent there.

We now show that S^+ and S^- must be unbounded since $\mathfrak{C}(z)$ is not constant. First, note that $\mathfrak{C}(z) = \overline{\mathfrak{C}(\bar{z}^{-1})}$. As in the Schwarz reflection principle, define $B = S^+ \cup (\bar{S}^+)^{-1}$. Clearly, $\Re\mathfrak{C}(z) = 0$ for $z \in \partial B$. If S^+ is bounded, then B is bounded as well. By the real max modulus principle, $\Re\mathfrak{C}(z)$ must be zero inside B, and hence $\Re\mathfrak{C}(z)$ is bounded everywhere, which is impossible.

Finally we show that the regions S^+ and S^- "fill out" to open sectors as $|z| \to \infty$. That is to say, if S is some sector in which $\Re z^N > 0$, then for any ray $\{re^{i\theta} : r > 1\}$ contained in S, there exists $R = R(\theta)$ so that the truncated ray $\{re^{i\theta} : r > R(\theta)\} \subset S^+$.

Without loss of generality[5], let us suppose that $C_N \in \mathbb{R}^+$. For very large $|z|$, we write $\mathfrak{C}(z) = \sum_{j=1}^N \bar{C}_j z^j + C_j z^{-j} = \bar{C}_N z^N + O(z^{N-1})$. Then setting $z = re^{i\theta}$, we find that $r^{-N}\mathfrak{C}(re^{i\theta}) = \bar{C}_N e^{iN\theta} + O(r^{-1})$. Thus, for r sufficiently large and $N\theta \neq (2m+1)\pi/2$, we find that $r^{-N}\mathfrak{C}(re^{i\theta})$ has either strictly positive real part or strictly negative real part. In particular, if $|N\theta \mp \pi/2| > \epsilon$, then there exists an $R = R(\epsilon, \theta)$ so that $\Re r^{-N}\mathfrak{C}(re^{i\theta})$ is bounded strictly away from zero.

Motivated by the above, we define the following subsets of $\mathbb{C}$ (with $j = 0 \ldots N-1$):

$$A^+_{j,\epsilon} = \{re^{i\theta} : r \geq R(\epsilon, \theta), \\ \theta \in [-\pi/2N + 2\pi j/N + \epsilon, \pi/2N + 2\pi j/N - \epsilon]\} \quad (2.13a)$$

$$A^-_{j,\epsilon} = \{re^{i\theta} : r \geq R(\epsilon, \theta), \\ \theta \in [-\pi/2N + 2\pi(j+1/2)/N + \epsilon, \pi/2N + 2\pi(j+1/2)/N - \epsilon]\} \quad (2.13b)$$

[5] Suppose $C_N = \rho e^{i\theta}$. Then rather than choosing $z = e^{i\omega t}$, we would substitute $z = e^{i(\omega t - \theta/N)}$.

Clearly, for sufficiently large R, $A^+_{j,\epsilon} \setminus B_R \subset S^+$ and $A^-_{j,\epsilon} \setminus B_R \subset S^-$. Here, B_R is the ball of radius R about $z = 0$.

Step 3: Asymptotics of $f(z)$

We now show that $f(z) = 0$. We begin by writing $f(z)$ as follows:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = -\sum_{n=1}^{\infty} g_n z^{-n} + \sum_{m<0} \psi^R_m z^m e^{-\lambda_m \mathfrak{C}(z)}, z \in S^+ \tag{2.14a}$$

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = -\sum_{n=1}^{\infty} g_n z^{-n} + \sum_{m<0} \psi^L_m z^m e^{+\lambda_m \mathfrak{C}(z)}, z \in S^- \tag{2.14b}$$

We let $S_k, k = 0, \ldots, 2N+1$ be a set of sectors of opening $\pi/(2N+1)$ arranged in such a way that the boundaries of S_k avoid the rays $re^{i\pi(2j+1)/2N}$. Therefore, for sufficiently large $|z|$, the boundaries of S_k are contained in either $A^+_{j,\epsilon}$ or $A^-_{j,\epsilon}$ except for a compact region. On ∂S_k, $f(z)$ is decaying as $|z| \to \infty$, by a simple examination of (2.14). Since $f(z)$ is entire (unlike $\psi(z)$), $f(z)$ is also bounded on ∂S_k even for small z.

We have shown that $f(z)$ is bounded on ∂S_k. Applying the Phragmen-Lindelöf theorem, $f(z)$ is therefore bounded on S_k. Since $\cup_{k=0}^{2n+1} S_k = \mathbb{C}$, we find $f(z)$ is constant. Since we know that along any ray contained in $A^{\pm}_{j,\varepsilon}$, $f(z)$ is decreasing, we know $f(z) = 0$.

Step 4: Asymptotics of $g(z)$

We now show that $g(z) = 0$. We rewrite (2.14) with $g(z)$ on the left side.

$$\sum_{n=1}^{\infty} g_n z^{-n} = \sum_{m<0} \psi^R_m z^m e^{-\lambda_m \mathfrak{C}(z)}, z \in S^+ \tag{2.15a}$$

$$\sum_{n=1}^{\infty} g_n z^{-n} = \sum_{m<0} \psi^L_m z^m e^{+\lambda_m \mathfrak{C}(z)}, z \in S^- \tag{2.15b}$$

Since the left sides of (2.15a) and (2.15b) are (convergent) asymptotic power series (for sufficiently large $|z|$), while the right sides of (2.15a) and (2.15b) are (convergent) asymptotic series of exponentials, we find that the right side decays much faster than the left side. This is impossible unless both sides are zero. $\square$

3 The Floquet Formulation

In this section we prove Theorem 2. To do so we consider the function $Y(t) = \psi(0,t)$ and derive a closed integral equation for it via Duhamel's formula. Computing the Zak transform in time of this equation yields an integral equation of compact Fredholm type for $\mathcal{Z}[Y](\sigma, t)$, the Zak transform of $Y(t)$. The integral operator is shown to be analytic in σ; the analytic Fredholm alternative to this equation shows that $\mathcal{Z}[Y](\sigma,t)$ is meromorphic[6] in $\sigma^{1/2}$ for $\Re\sigma \in [0,\omega)$. The poles corresponds to resonances or bound states, while the branch point at $\sigma = 0$ corresponds to the dispersive part of the solution.

In Section 3.3 we extend these results from $x = 0$ to the entire real line. We show that the wavefunction can be decomposed in the form (1.6). If $\Im\check{\sigma}_k = 0$, then $\Re\check{\sigma}_k \in (0,\omega)$ and $\Phi_{k,0}(x,t)$ corresponds to a Floquet bound state. The remainder $\Psi_M(x,t)$ decays with time, in particular $|\Psi_M(x,t)| = O(t^{-1/2})$ (or faster) as $t \to \infty$, though not uniformly in x.

3.1 Setting up the problem

We work in the velocity gauge. We rewrite (1.9) in Duhamel form, using the Green's function for the free Schrödinger equation, $(4\pi i t)^{-1/2}e^{ix^2/4t}$:

$$\begin{aligned}\psi_v(x,t) &= \psi_{v,0}(x,t)\\ &+ 2i\int_0^t\int_{\mathbb{R}} \exp\left(\frac{i(x-x')^2}{4(t-t')}\right)\delta(x'-c(t'))\psi_v(x',t')dx'\frac{dt'}{\sqrt{4\pi i(t-t')}}\end{aligned} \tag{3.1}$$

where we have defined:

$$\psi_{v,0}(x,t) = e^{i\partial_x^2 t}\psi_v(x,0) = \int_{\mathbb{R}}(4\pi i t)^{-1/2}e^{i\left|x-x'\right|^2/4t}\psi_v(x',0)dx'$$

Computing the x' integral explicitly and changing variables to $s = t - t'$ yields:

$$\begin{aligned}\psi_v(x,t) &= \psi_{v,0}(x,t)\\ &+ 2i\int_0^t \exp\left(\frac{i(x-c(t-s))^2}{4s}\right)\psi_v(c(t-s),t-s)\frac{ds}{\sqrt{4\pi i s}}\end{aligned} \tag{3.2}$$

We now substitute $x = c(t)$, to obtain a closed equation for $\psi_v(c(t),t)$:

$$\begin{aligned}\psi_v(c(t),t) &= \psi_{v,0}(c(t),t)\\ &+ \sqrt{\frac{i}{\pi}}\int_0^t \exp\left(\frac{i(c(t)-c(t-s))^2}{4s}\right)\psi_v(c(t-s),t-s)\frac{ds}{\sqrt{s}}\end{aligned} \tag{3.3}$$

Letting $Y_0(t) = \psi_{v,0}(c(t),t)$ and $Y(t) = \psi(c(t),t)$ for $t \geq 0$ (both are set equal to 0 for $t < 0$) we obtain:

$$Y(t) = Y_0(t) + \sqrt{\frac{i}{\pi}}\int_0^t \exp\left(\frac{i(c(t)-c(t-s))^2}{4s}\right)Y(t-s)\frac{ds}{\sqrt{s}} \tag{3.4}$$

The main tool of our analysis will be the Zak transform.

[6] The branch point at $\sigma = 0$ is repeated at $\sigma = n\omega$, $n \in \mathbb{Z}$, due to the pseudo-periodicity of the Zak transform, c.f. (3.6c). This also makes it sufficient to consider only $\Re\sigma \in [0,\omega)$.

Definition 3.1 *Let $f(t) = 0$ for $t < 0$ and $|f(t)| \leq Ce^{\alpha t}$ for some $\alpha > 0$. Then $f(t)$ is said to be Zak transformable. The Zak transform of $f(t)$ is defined (for $\Im\sigma > \alpha$) by:*

$$\mathcal{Z}[f](\sigma, t) = \sum_{j \in \mathbb{Z}} e^{i\sigma(t+2\pi j/\omega)} f(t + 2\pi j/\omega) \tag{3.5}$$

and by the analytic continuation of (3.5) when $\Im\sigma < \alpha$, provided that the analytic continuation exists (treating $\mathcal{Z}[f](\sigma, t)$ as a function of σ taking values in $L^2([0, 2\pi/\omega], dt)$).

Proposition 3.2 *$\mathcal{Z}[f](\sigma, t)$ has the following properties:*

$$f(t) = \omega^{-1} \int_{i\beta}^{i\beta+\omega} e^{-i\sigma t} \mathcal{Z}[f](\sigma, t) d\sigma \tag{3.6a}$$

If $\mathcal{Z}[f](\sigma, t)$ is singular for $\Im\sigma = \beta$, this integral is interpreted as the limit of integrals over the contours $[i(\beta + \epsilon), i(\beta + \epsilon) + \omega]$ as $\epsilon \to 0$ from above.

$$\mathcal{Z}[f](\sigma, t + 2\pi/\omega) = \mathcal{Z}[f](\sigma, t) \tag{3.6b}$$

$$\mathcal{Z}[f](\sigma + \omega, t) = e^{i\omega t} \mathcal{Z}[f](\sigma, t) \tag{3.6c}$$

If $p(t)$ is $2\pi/\omega$-periodic, then:

$$\mathcal{Z}[pf](\sigma, t) = p(t)\mathcal{Z}[f](\sigma, t) \tag{3.6d}$$

With the exception of (3.6a), these results all follow immediately from (3.5). See Remark 3.5 for an explanation of (3.6a).

Remark 3.3 Suppose $f(t)$ is Zak transformable, and uniformly bounded in time ($\alpha = 0$). Suppose further that the analytic continuation of $\mathcal{Z}[f](\sigma, t)$ has a singularity (say at $\sigma = 0$). Then (3.6c) still holds, in the sense that for any direction θ, $\mathcal{Z}[f](\sigma + \omega + 0e^{i\theta}, t) = e^{i\omega t}\mathcal{Z}[f](\sigma + 0e^{i\theta}, t)$.

Remark 3.4 More information on the Zak transform can be found in, e.g., [12, p.p. 109-110]. Our definition differs slightly from that in [12] by allowing σ to take complex values.

Remark 3.5 One can relate the Zak and Fourier transforms as follows. Let $\hat{f}(k) = \int e^{ikt} f(t) dt$ denote the Fourier transform of $f(t)$. Then:

$$\mathcal{Z}[f](\sigma, t) = \frac{\omega}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(\sigma + n\omega) e^{-in\omega t} \tag{3.7}$$

The Poisson summation formula, applied to (3.5), yields (3.7). Eq. (3.6a) follows immediately from (3.7). This relation implies that our approach is equivalent to the Fourier/Laplace transform analysis done in [7, 8, 1]. The Zak transform is used simply for algebraic convenience.

We proceed as follows. Applying the Zak transform to (3.4) yields an integral equation of the form

$$y(\sigma,t) = y_0(\sigma,t) + K(\sigma)y(\sigma,t) \tag{3.8}$$

with $y(\sigma,t) = \mathcal{Z}[Y](\sigma,t)$, $y_0(\sigma,t) = \mathcal{Z}[Y_0](\sigma,t)$ and $K(\sigma)$ the Zak transform of the integral operator in (3.4). $K(\sigma)$ will be shown to be meromorphic in σ as a compact operator family from $L^2(S^1,dt) \to L^2(S^1,dt)$, except for a branch point at $\sigma = 0$.

We then use the Fredholm alternative theorem to invert $(1-K(\sigma))$. Once this is done, we find:

$$y(\sigma,t) = (1-K(\sigma))^{-1}y_0(\sigma,t) \tag{3.9}$$

The poles of $(1-K(\sigma))^{-1}$ correspond to resonances, and a branch point at $\sigma = 0$ corresponds to the dispersive part of the solution, i.e. the part with polynomial decay in $t^{-1/2}$.

To begin, we determine the analyticity properties of $\mathcal{Z}[Y_0](\sigma,t)$.

Proposition 3.6 *Suppose $\psi_0(x)$ is smooth and compactly supported. Then near $\sigma = 0$, $y_0(\sigma,t)$ has the expansion:*

$$\mathcal{Z}[Y_0](\sigma,t) = y_0(\sigma,t) = \frac{1}{2}\sigma^{-1/2}\int_{\mathbb{R}} \psi_0(x)dx + f(\sigma,t) \tag{3.10}$$

The function $f(\sigma,t)$ is analytic in $\sigma^{1/2}$ for $\Re\sigma \in [0,\omega)$ (there are similar branch points at $\sigma = n\omega$), and is in $L^2(S^1,dt)$ for each σ. Also, for some constants C_1 and C_2, we have

$$\|\mathcal{Z}[Y_0](\sigma,t)\|_{L^2(S^1_\omega,dt)} \le C_1 e^{C_2|\Im\sigma|}$$

Here, $S^1_\omega = \mathbb{R}/(2\pi/\omega)\mathbb{Z}$ is the set $[0,2\pi/\omega]$ with periodic boundaries.

The same conclusion follows for $\mathcal{Z}[\psi(x+c(t),t)](\sigma,t)$ for any fixed x.

Proof. We can write out $Y_0(t)$ using the Fourier transform as:

$$Y_0(t) = \chi_{\mathbb{R}^+}(t)\int_{\mathbb{R}} e^{ikc(t)}e^{ik^2t}\hat{\psi}_0(k)dk$$

Computing the Zak transform yields:

$$
\begin{aligned}
\mathcal{Z}[Y_0](\sigma,t) &= \sum_{j\in\mathbb{Z}} e^{i\sigma(t-2\pi j/\omega)}\chi_{\mathbb{R}^+}(t-2\pi j/\omega)\int_{\mathbb{R}} e^{ikc(t)}e^{ik^2(t-2\pi j/\omega)}\hat{\psi}_0(k)dk \\
&= \int_{\mathbb{R}} e^{ikc(t)}\hat{\psi}_0(k)\left[\sum_{j\in\mathbb{Z}} e^{i\sigma(t-2\pi j/\omega)}e^{ik^2(t-2\pi j/\omega)}\chi_{\mathbb{R}^+}(t-2\pi j/\omega)\right]dk \\
&= \int_{\mathbb{R}} e^{ikc(t)}\hat{\psi}_0(k)\left[\sum_{n\in\mathbb{Z}} \frac{e^{-in\omega t}}{i(k^2+\sigma+n\omega)}\right]dk \\
&= \sigma^{-1/2}(1/2)\int_{\mathbb{R}}\psi_0(y)dy + \sigma^{-1/2}(1/2)\int_{\mathbb{R}}(e^{-\sqrt{\sigma}|c(t)-y|}-1)\psi_0(y)dy \\
&\quad + \sum_{n\neq 0}\frac{e^{-in\omega t}}{2\sqrt{\sigma+n\omega}}\int_{\mathbb{R}} e^{-\sqrt{\sigma+n\omega}|c(t)-y|}\psi_0(y)dy \qquad (3.11)
\end{aligned}
$$

The interchange of the sum and integral between lines 1 and 2 is justified (for $\Im\sigma > 0$ and t fixed) since the sum over j is absolutely convergent, as the integral over k is bounded and uniformly convergent with respect to k.

The change inside the square brackets between lines 2 and 3 comes from the Poisson summation formula in the t variable, and the fact that the Fourier transform of $\chi_{\mathbb{R}^+}(t)e^{i(k^2+\sigma)t}$ is $-i(k^2+\sigma+\zeta)^{-1}$(with ζ dual to t).

The first term on the right side of (3.11) agrees with that in (3.10). Since $(e^{-\sqrt{\sigma}|c(t)-y|}-1)$ is analytic in $\sigma^{1/2}$ and has no constant term, the second term is analytic in $\sigma^{1/2}$. The third term is analytic in σ. When added together, these terms become $f(\sigma,t)$ which is analytic in $\sigma^{1/2}$. The result is valid for arbitrary σ by analytic continuation.

Since $\psi_0(x)$ is supported on a compact region, $|c(t)-y|$ is bounded (say by C_2) and exponential growth follows. The result follows for all x by translation invariance of $e^{i\partial_x^2 t}$. □

We now determine the Zak transform of the integral operator in (3.4) and compute the resolvent of it.

3.2 Construction of the resolvent

We now apply the Zak transform to (3.4) to construct an equivalent integral equation.

Proposition 3.7 *Let $f(t)$ be Zak transformable. Consider the integral operator:*

$$K_V f(t) = \sqrt{\frac{i}{\pi}}\int_0^t \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right)f(t-s)\frac{ds}{\sqrt{s}} \qquad (3.12)$$

For $\Im\sigma > 0$, define $\mathcal{Z}[K_V f](\sigma,t) = (K(\sigma)\mathcal{Z}[f])(\sigma,t)$. Then:

$$K(\sigma)f(\sigma,t) = \sqrt{\frac{i}{\pi}}\int_0^\infty \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right)e^{i\sigma s}\mathcal{Z}[f](\sigma,t-s)\frac{ds}{\sqrt{s}} \qquad (3.13)$$

For $\Im\sigma \leq 0$, we define $K(\sigma)$ to be the analytic continuation of $K(\sigma)$ (where it exists).

Proof. We rewrite (3.12) as:

$$\sqrt{\frac{i}{\pi}} \int_0^t \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) f(t-s)\frac{ds}{\sqrt{s}} = \sqrt{\frac{i}{\pi}} \int_{\mathbb{R}} \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) f(t-s)\chi_{\mathbb{R}^+}(s)\frac{ds}{\sqrt{s}} \quad (3.14)$$

Applying $\mathcal{Z}$ to both sides of (3.14) yields

$$\mathcal{Z}[K_V f](\sigma,t) = \sum_{j\in\mathbb{Z}} e^{i\sigma(t+2\pi j/\omega)}[K_V f](t+2\pi j/\omega)$$

$$= \sqrt{\frac{i}{\pi}} \sum_{j\in\mathbb{Z}} e^{i\sigma(t+2\pi j/\omega)} \int_{\mathbb{R}} \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) f(t+2\pi j/\omega - s)\chi_{\mathbb{R}^+}(s)\frac{ds}{\sqrt{s}}$$

$$= \sqrt{\frac{i}{\pi}} \int_{\mathbb{R}} \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) e^{i\sigma s} \times \left[\sum_{j\in\mathbb{Z}} e^{i\sigma(t-s+2\pi j/\omega)} f(t-s+2\pi j/\omega)\right] \chi_{\mathbb{R}^+}(s)\frac{ds}{\sqrt{s}}$$

$$= \sqrt{\frac{i}{\pi}} \int_0^\infty \exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) e^{i\sigma s}\mathcal{Z}[f](\sigma, t-s)\frac{ds}{\sqrt{s}} \quad (3.15)$$

This is what we wanted to show. □

We now show that the operator $K(\sigma)$, constructed above, is compact. We decompose $K(\sigma)$ as $K_F(\sigma) + K_L(\sigma)$ (defined shortly), and treat each piece separately.

Proposition 3.8 *Define $K_F(\sigma) : L^2(S^1, dt) \to L^2(S^1, dt)$ by:*

$$K_F(\sigma)f(t) = \sqrt{\frac{i}{\pi}} \int_0^\infty e^{i\sigma s} f(t-s)\frac{ds}{\sqrt{s}}$$

Then, $K_F(\sigma)$ is compact and analytic for $\Im\sigma > 0$. It can be analytically continued to $\Im\sigma \leq 0$, $\sigma \neq 0$, and the continuation has a $\sigma^{-1/2}$ branch point at $\sigma = 0$.

Proof. We compute this exactly by expanding $f(t)$ in Fourier series and interchanging the order of summation and integration:

$$\sqrt{\frac{i}{\pi}} \sum_{n\in\mathbb{Z}} f_n e^{-in\omega t} \int_0^\infty e^{i(\sigma+n\omega)s}\frac{ds}{\sqrt{s}} = \sum_{n\in\mathbb{Z}} \frac{f_n}{\sqrt{\sigma+n\omega}} e^{-in\omega t} \quad (3.16)$$

This is valid for $\Im\sigma > 0$, as well as $\Im\sigma = 0$ but in this case we must treat the integral as improper.

Thus, in the basis $e^{-in\omega t}$, this operator is diagonal multiplication by $(\sigma + n\omega)^{-1/2}$. Compactness follows since the diagonal elements decay in both directions. Analyticity for $\sigma \neq 0$ follows by inspection of the right side of (3.16), and choosing the branch cut of $\sqrt{\sigma + n\omega}$ to lie on the negative real line. □

Proposition 3.9 *Define* $K_L(\sigma) : L^2(S^1, dt) \to L^2(S^1, dt)$ *as:*

$$K_L(\sigma)f(t) = \sqrt{\frac{i}{\pi}} \int_0^\infty \left[\exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) - 1\right] e^{i\sigma s} f(t-s)\frac{ds}{\sqrt{s}} \quad (3.17)$$

Then $K_L(\sigma)$ *is compact for* $\Im\sigma \geq 0$ *and analytic for* $\Im\sigma > 0$. *It has continuous limiting values at* $\Im\sigma = 0$.

Proof. We rewrite (3.17) as:

$$\int_0^{2\pi/\omega} \sum_{k=0}^{\infty} \left[\exp\left(i\frac{(c(t)-c(t-s))^2}{4(s+2\pi k/\omega)}\right) - 1\right] \frac{e^{i\sigma(s+2\pi k/\omega)}}{\sqrt{s+2\pi k/\omega}} f(t-s)ds \quad (3.18)$$

If $\Im\sigma \geq 0$, the summands decay at least as fast as $k^{-3/2}$ as $k \to \infty$. Each term in the sum is continuous. Thus the sum is absolutely convergent to a smooth function in t and s, which is analytic in σ (thus the limit is analytic except when $\Im\sigma = 0$). The region of integration is compact, and so is $K_L(\sigma)$. □

We now analytically continue $K_L(\sigma)$ to the strip $0 < \Re\sigma < \omega$.

Proposition 3.10 *Let* $K'(\sigma)$ *be the integral operator defined by:*

$$K'(\sigma)f(t) = \int_0^{2\pi/\omega} k'_\sigma(t,s)f(t-s)ds \quad (3.19a)$$

$$k'_\sigma(t,s) = \frac{\omega}{2\pi i}\int_{\mathcal{C}} \frac{e^{\sigma p}}{1-e^{\omega p - i\omega s}} \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right) - 1\right] \frac{dp}{\sqrt{p}} \quad (3.19b)$$

where $\mathcal{C}$ *is some contour along the real line in the upper half plane which avoids the singularities of the integrand at* $p = 0$ *and* $p = i(s + 2\pi n/\omega)$ *(see the proof for a specific example).*

Then $K'(\sigma)$ *is an analytic (in* σ*) family of compact operators for* $0 < \Re\sigma < \omega$, *and vanishes as* $\Im\sigma \to +\infty$. *Furthermore,* $K'(\sigma)$ *is the analytic continuation of* $K_L(\sigma)$. *Finally, for* $\sigma = -i\lambda$ *(with* $\Re\lambda < 0$*) or* $\sigma = -i\lambda + \omega$, $K(\sigma)$ *is analytic in the parameter* $\lambda^{1/2}$ *or* $(\sigma - \omega)^{1/2}$.

Proof.

Step 1: Analyticity

To perform the integral in (3.19b), we let $\gamma_R(t) = t\omega R/2\pi$ for $t \in \mathbb{R} \setminus [-2\pi/\omega, 2\pi/\omega]$, and $\gamma_R(t) = Re^{i[\pi-(\omega t+2\pi)/4]}$ for $t \in [-2\pi/\omega, 2\pi/\omega]$. That is,

$\gamma_R(t)$ travels along the real line, and circles upward around the disk of radius R. The integral is then defined as $\lim_{R\to 0}\int_{\gamma_R}(\cdot)\,dp$.

To compute the behavior of the integral, simply take $R = 2\pi/\omega$:

$$\begin{aligned}\omega^{-1}k'_\sigma(t,s) &= \int_{\mathbb{R}+0i} \frac{e^{\sigma p}}{1-e^{\omega p - i\omega s}}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right)-1\right]\frac{dp}{\sqrt{p}} \\ &= \int_{\gamma} \frac{e^{\sigma p}}{1-e^{\omega p - i\omega s}}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right)-1\right]\frac{dp}{\sqrt{p}} \\ &\qquad + \frac{2\pi i}{\omega}e^{i\sigma s}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4is}\right)-1\right]\frac{1}{\sqrt{is}} \end{aligned} \tag{3.20}$$

The integrand in the first term is analytic in t and s since p stays away from 0 (thus avoiding the essential singularity at $p=0$). It is exponentially decaying both for large positive p (at the rate $e^{(\sigma-\omega)p}$) and for large negative p (at the rate $e^{-\sigma p}$). If $\Re\sigma = 0$ or $\Re\sigma = \omega$, the integrand still decays at the rate $p^{-3/2}$, which is integrable.

The last term is integrable at $s = 0$ and analytic (in s) elsewhere. Thus, $k'_\sigma(t,s)$ has only a singularity of order $s^{-1/2}$, and is analytic elsewhere. This shows that $K'(\sigma)$ is a compact family of operators, analytic on σ.

Step 2: Vanishing of the operator as $\Im\sigma \to +\infty$

We examine (3.20). The first term vanishes as $\Im\sigma \to \infty$ by the Riemann-Lebesgue lemma. The second term vanishes since $e^{i\sigma s}$ does. Thus, $k'_\sigma(t,s)\to 0$, and so does $K'(\sigma)$.

Step 3: Continuation of $K_L(\sigma)$

To show that $K'(\sigma) = K_L(\sigma)$ if $\Im\sigma > 0$, we simply move the contour of integration in (3.19b) upward and collect residues:

$$\begin{aligned}&\int_{\mathbb{R}+0i} \frac{e^{\sigma p}}{1-e^{\omega p - i\omega s}}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right)-1\right]\frac{dp}{\sqrt{p}} \\ &\quad = \lim_{N\to\infty}\left[\int_{\mathbb{R}+i2\pi N/\omega} \frac{e^{\sigma p}}{1-e^{\omega p - i\omega s}}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right)-1\right]\frac{dp}{\sqrt{p}}\right. \\ &\quad \left. + \sum_{j=0}^{N}\frac{2\pi i}{\omega}e^{i\sigma(s+2\pi j/\omega)}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4i(s+2\pi j/\omega)}\right)-1\right]\frac{1}{\sqrt{i(s+2\pi j/\omega)}}\right] \\ &\quad = \sum_{j=0}^{\infty}\frac{2\pi i}{\omega}e^{i\sigma(s+2\pi j/\omega)}\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4i(s+2\pi j/\omega)}\right)-1\right]\frac{1}{\sqrt{i(s+2\pi j/\omega}}\end{aligned}$$

We then integrate this kernel against an $L^2(S^1, dt)$ function $f(t)$ and obtain:

$$\int_0^{2\pi/\omega} \sqrt{\frac{i}{\pi}} \frac{\omega}{2\pi i} \sum_{j=0}^{\infty} \frac{2\pi i}{\omega} e^{i\sigma(s+2\pi j/\omega)}$$
$$\times \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4i(s+2\pi j/\omega)}\right) - 1\right] \frac{1}{\sqrt{i(s+2\pi j/\omega)}} f(t-s)ds$$
$$= \sqrt{\frac{i}{\pi}} \int_0^{\infty} \left[\exp\left(i\frac{(c(t)-c(t-s))^2}{4s}\right) - 1\right] e^{i\sigma s} f(t-s)\frac{ds}{\sqrt{s}}$$

This is in agreement with (3.17). Hence, $K'(\sigma) = K_L(\sigma)$ for $\Im\sigma > 0$, $\Re\sigma \in (0, \omega)$ and therefore $K'(\sigma)$ is the analytic continuation of $K_L(\sigma)$.

Step 4: Singularity at $\sigma = 0, \omega$

We now wish to show that $K'(-i\lambda)$ is analytic in $\sqrt{\lambda}$ for $\sigma = -i\lambda$, and similarly that $K(-i\lambda + \omega)$ is analytic in λ. To do this, we proceed as in Step 3, but push the contour down instead of up. We rotate the contour $\gamma_1 \cup \gamma_2 \cup \gamma_3$, with $\gamma_1 = [-i\infty - R, -R]$, γ_2 which goes around the unit circle of radius R in the upper half plane (as in step 1), and γ_3 which is $[R, R - i\infty]$. This lets us avoid concerning ourselves with the singularities of the integrand; the important behavior is the decay near $p = -i\infty$.

Note that the integral kernel of $K'(-i\lambda)$ is given by

$$k'_{-i\lambda}(t,s) = \int_{\gamma_1\cup\gamma_2\cup\gamma_3} \frac{e^{-i\lambda p}}{1-e^{\omega p - i\omega s}} \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right) - 1\right] \frac{dp}{\sqrt{p}}$$

while that of $K'(-i\lambda + \omega)$ is given by:

$$k'_{-i\lambda+\omega}(t,s) = \int_{\gamma_1\cup\gamma_2\cup\gamma_3} \frac{e^{-i\lambda p}e^{\omega p}}{1-e^{\omega p - i\omega s}} \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right) - 1\right] \frac{dp}{\sqrt{p}}$$

First, observe that the integral over γ_2 is analytic in λ, provided $R \neq \omega s$. Thus, choosing a different R for $\omega s < (3/4)\pi$ and $\omega s > (1/4)\pi$ shows analyticity in λ.

We consider the case $\sigma = -i\lambda$, the case $\sigma = -i\lambda + \omega$ being treated similarly. We now observe that, for $\Re p = R$ (the same argument applies to $\Re p = -R$), the integrand (over γ_3 or γ_1) becomes a Laplace transform:

$$\int_{\gamma_3} \frac{e^{-i\lambda p}}{1-e^{\omega p - i\omega s}} \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4p}\right) - 1\right] \frac{dp}{\sqrt{p}}$$
$$= e^{-i\lambda R} \int_0^{-i\infty} \frac{e^{-i\lambda p}}{1-e^{\omega(p+R) - i\omega s}} \left[\exp\left(\frac{(c(t)-c(t-s))^2}{4(p+R)}\right) - 1\right] \frac{dp}{\sqrt{p+R}} \tag{3.21}$$

We then observe that we can rewrite

$$\left[\exp\left(\frac{(c(t)-c(t-s))^2}{4(p+R)}\right) - 1\right] \frac{1}{\sqrt{p+R}} = (p+R)^{-3/2} H(c(t), c(t-s), p+R)$$

with $H(c(t), c(t-s), p+R)$ analytic in p for $p \neq R$, and therefore analytic on the contour in (3.21). This follows since $e^z - 1 = O(z)$ near $z = 0$. We now substitute this back into (3.21) and change variables to $i\lambda p = z$, to obtain:

$$
\begin{aligned}
(3.21) &= -ie^{-i\lambda R} \int_0^\infty e^{-z} \frac{1}{1-e^{\omega(-iz/\lambda+R)-i\omega s}} \frac{H(c(t), c(t-s), -iz/\lambda + R)}{(-iz/\lambda+R)^{3/2}} \frac{dz}{\lambda} \\
&= -i\lambda^{1/2} e^{-i\lambda R} \int_0^\infty e^{-z} \frac{1}{1-e^{\omega(-iz/\lambda+R)-i\omega s}} \frac{H(c(t), c(t-s), -iz/\lambda + R)}{(-iz+R\lambda)^{3/2}} dz
\end{aligned}
\tag{3.22}
$$

The integrand is analytic in λ, and absolutely convergent. The power of $\lambda^{1/2}$ makes the net result a ramified analytic function. The same argument can be applied to γ_1, replacing R by $-R$. Thus, we have shown that $k'_{-i\lambda}(t,s)$ is analytic in $\lambda^{1/2}$. This implies that $K(-i\lambda)$ is analytic in $\lambda^{1/2}$. As remarked before, the case $K(-i\lambda+\omega)$ is identical, so the proof is complete. □

Now we have shown that $K' = K_L$. In addition, now that $K_L(\sigma)$ and $K_F(\sigma)$ are defined, it is clear that $K_F(\sigma) + K_L(\sigma) = K(\sigma)$. Thus, $K(\sigma) = K_F(\sigma) + K_L(\sigma)$ can be analytically continued to the region $\Im\sigma \leq 0$. Next we show that $K(\sigma)$ grows at most exponentially as $\Im\sigma \to \pm\infty$.

Proposition 3.11 *$K(\sigma)$ vanishes as $\Im\sigma \to \infty$.*

Proof. We break $K(\sigma)$ up as $K(\sigma) = K_F(\sigma) + K_L(\sigma)$. The first term, $K_F(\sigma)$ is bounded (away from $\sigma = 0$) simply by inspecting (3.16). The second vanishes near $\Im\sigma = \infty$ by Proposition 3.10. □

We have now shown that $K(\sigma) : L^2(S^1_\omega, dt) \to L^2(S^1_\omega, dt)$ is an analytic (in σ) family of compact operators. This allows us to construct the resolvent.

Proposition 3.12 *The operator $(1-K(\sigma))^{-1}$ is a meromorphic (in σ) family of bounded operators. This implies that if $(1-K(\sigma))^{-1}$ has a pole of order n at a point $\sigma = \breve{\sigma}_k$, we then have the following asymptotic expansion as $\sigma \to \breve{\sigma}_k$:*

$$
(1-K(\sigma))^{-1} = \sum_{j=0}^{n_k} \frac{Y_{k,j}(t)\langle Y_{k,j}(t)|\cdot\rangle}{(\sigma-\breve{\sigma}_k)^{j+1}} + D(\sigma) \tag{3.23}
$$

where $D(\sigma)$ is analytic near $\breve{\sigma}_k$. $Y_{k,j}(t)$ solves $(1-K(\breve{\sigma}_k))Y_{k,j}(t) = Y_{k,j-1}(t)$ (with $Y_{k,-1}(t) = 0$). The functions $Y_{k,j}(t)$ are all $L^2(S^1_\omega)$ functions.

If $\breve{\sigma}_k = 0$, then the same result holds, except that the poles are in the variable $\sqrt{\sigma}$ instead of $(\sigma - \breve{\sigma}_k)$.

An additional result (which we use later) is that if $(1-K(\sigma))^{-1}$ has no poles in the variable $\sigma^{1/2}$ near $\sigma = 0$, then $P_0 y(0,t) = (1/2)\int_{\mathbb{R}} \psi_0(x)dx$, where P_0 is projection onto the zero'th Fourier coefficient.

Remark 3.13 Note that we do not assume that dim Ker $K(\breve{\sigma}_k) \neq 1$. We choose the convention that if the dimension of the kernel is greater than 1, we consider this to mean that $\breve{\sigma}_k = \breve{\sigma}_{k'}$ for some $k \neq k'$. Thus, one can have degenerate resonances as well as degenerate eigenvalues.

Proof. This is merely the analytic Fredholm alternative theorem. There is only one technical point regarding the behavior near $\sigma = 0$ due to the fact that $K(\sigma)$ is singular there.

This can be remedied as follows. Note that $K(\sigma) = K_F(\sigma) + K_L(\sigma)$. For small σ, we use the following resolvent identity:

$$\begin{aligned}(1 - K(\sigma)^{-1} &= (1 - K_F(\sigma) - K_L(\sigma))^{-1} \\ &= [1 - K_F(\sigma)]^{-1}(1 - K_L(\sigma)[1 - K_F(\sigma)]^{-1})^{-1} \end{aligned} \tag{3.24}$$

The operator $[1 - K_F(\sigma)]^{-1}$ is analytic in $\sigma^{1/2}$ (for small σ), being defined by

$$[1 - K_f(\sigma)]^{-1} \sum_j f_j e^{-ij\omega t} = \sum_j (1 - \sigma^{-1/2})^{-1} f_j e^{-ij\omega t}.$$

Since $K_L(\sigma)$ is compact, the Fredholm alternative applies to the resolvent of $(1 - K_L(\sigma)[1 - K_F(\sigma)]^{-1})$, implying that $(1 - K_L(\sigma)[1 - K_F(\sigma)]^{-1})^{-1}$ and therefore $(1 - K(\sigma))^{-1}$ is meromorphic in $\sigma^{1/2}$.

If this operator has no poles in the variable $\sigma^{1/2}$ near $\sigma = 0$, then $y(\sigma)$ is analytic in $\sigma^{1/2}$ for small σ. We then rewrite (3.8) as:

$$\begin{aligned}(1 - K_F(\sigma)(1 - P_0) - K_L(\sigma))y(\sigma, t) + \sigma^{-1/2} P_0 y(\sigma, t) \\ = \sigma^{-1/2}(1/2) \int_{\mathbb{R}} \psi_0(x) dx + f(\sigma^{1/2}, t)\end{aligned}$$

Matching coefficients to to order $\sigma^{-1/2}$ as $\sigma \to 0$ shows that $P_0 y(0, t) = (1/2) \int_{\mathbb{R}} \psi_0(x) dx$. □

Proposition 3.14 *Define $K_\epsilon(\sigma)$ as $K(\sigma)$ with $c(t)$ replaced by $\epsilon c(t)$ (so in particular, $K_1(\sigma) = K(\sigma)$). Then the position of the poles of $K_\epsilon(\sigma)$ are ramified analytic functions of ϵ, the field strength, except possibly near $\breve{\sigma}_k = -i\infty$. For small ϵ, there is only one pole $\breve{\sigma}_0$ near the real axis (corresponding to the dressed bound state), and all other poles are located near $\sigma = -i\infty$.*

Proof. This is basically the analytic implicit function theorem, using the fact that $K_\epsilon(\sigma)$ is analytic in ϵ, and $K_0(\sigma) = K_F(\sigma)$ (c.f. Proposition 3.8).

We first show that no poles form spontaneously. Consider a compact set, bounded by the curve γ. Then define

$$R_{\gamma,\epsilon} = \int_\gamma [1 - K_\epsilon(\sigma)]^{-1} d\sigma$$

Provided $[1 - K_\epsilon(\sigma)]^{-1}$ is analytic on γ (in σ and ϵ jointly), then $R_{\gamma,\epsilon}$ is analytic in ϵ. For $\epsilon = 0$, we find that:

$$[1 - K_0(\sigma)]^{-1} \sum_n f_n e^{-in\omega t} = \sum_n [1 - (\sigma + n\omega)^{-1/2}]^{-1} f_n e^{-in\omega t} \tag{3.25}$$

which has one pole on $[0,\omega)$, and no others. Since $\|K_\epsilon(\sigma) - K_0(\sigma)\| \leq C\epsilon$ (with C depending in γ and σ), $[1 - K_\epsilon(\sigma)]^{-1}$ is analytic on and inside γ for small ϵ. Thus $R_{\gamma,\epsilon} = 0$ for small ϵ, and by analyticity is zero on any interval $\epsilon \in [0,\delta)$ on which it is analytic. $R_{\gamma,\epsilon}$ is analytic as ϵ varies from $\epsilon = 0$, until $[1 - K_\epsilon(\sigma)]^{-1}$ becomes singular on γ. We therefore find that $R_{\gamma,\epsilon}$ becomes nonzero only after poles of $K_\epsilon(\sigma)$ have crossed γ, i.e. no poles formed spontaneously inside γ.

The same argument shows that spontaneous poles of higher order do not form if we use $R^k_{\gamma,\epsilon} = \int_\gamma f_k(\sigma)[1-K_\epsilon(\sigma)]^{-1}d\sigma$ (for $f_k(\sigma)$ a function with nonvanishing k-th derivative) instead of $R_{\gamma,\epsilon}$. This argument implies that any poles which are not present for $\epsilon = 0$ must come from $\sigma = -i\infty$ as ϵ is "switched on".

Analyticity of $\check{\sigma}_k$ follows immediately from Theorems 1.7 and 1.8 in [23, page 368-370] (see also the discussion following Theorem 1.7). These results show that any eigenvalue $\lambda(\epsilon,\sigma)$ of $K_\epsilon(\sigma)$ is an analytic function. Poles occur where $\lambda(\epsilon,\sigma) = 1$. By the implicit function theorem, $\check{\sigma}_k = \check{\sigma}_k(\epsilon)$ is ramified analytic. □

Remark 3.15 The only obstacle to proving the absence of poles moving in from $\sigma = -i\infty$ is a lack of bounds on the norm of $[1 - K(\sigma)]^{-1}$. If we had such bounds, it would be possible to show that the only pole of $[1 - K(\sigma)]^{-1}$ is the the analytic continuation of the bound state for $E(t) = 0$. In other cases we have considered [8, 9] such bounds were proved, and there is no fundamental reason it should not be true in this case as well.

3.3 Time behavior of $\psi(x,t)$

We have now shown that $K(\sigma)$ is a compact analytic operator. By the Fredholm alternative, $(1 - K(\sigma))^{-1}$ is a meromorphic operator family. By deforming the contour in (3.6a), we can determine the behavior of $Y(t)$. Once this is complete, we can calculate $\Phi_{k,j}(x,t)$ and $\Psi_M(x,t)$ and finish the proof of Theorem 2.

Proposition 3.16 *The function $Y(t)$ has the expansion:*

$$Y(t) = \sum_{k=0}^{M-1} \sum_{j=0}^{n_k} \alpha_{k,j} t^j e^{-i\check{\sigma}_k t} Y_{k,j}(t) + D_M(t) \tag{3.26}$$

with $Y_{k,j}(t)$ the residue of $[1-K(\sigma)]^{-1}$ at $\check{\sigma}_k$ and $\alpha_{j,k} = (2\pi/\omega)\langle y_0(\check{\sigma}_k,t)|Y_{k,j}(t)\rangle/j!$. M must not be greater than the number of poles of $[1 - K(\sigma)]^{-1}$. If no $\check{\sigma}_k = 0$, then $D_M(t)$ has the asymptotic expansion:

$$D_M(t) \sim \sum_{n\in\mathbb{Z}} e^{-in\omega t} \sum_{j=3}^{\infty} D_{j,n} t^{-j/2} \tag{3.27}$$

In particular this shows that $|D_M(t)| = O(t^{-3/2})$. If $\check{\sigma}_k = 0$ for some k, $Y(t) = D_M(t)$ except that in (3.27) the sum starts at $j = 1$ rather than $j = 3$.

Proof. Because $(1-K(\sigma))^{-1}$ is meromorphic in σ, $y(\sigma,t)$ can be written as

$$y(\sigma,t)=(1-K(\sigma))^{-1}y_0(\sigma,t)=\sum_{j=0}^{n_k}\frac{Y_{k,j}(t)\langle Y_{k,j}(t)|y_0(\sigma,t)\rangle}{(\sigma-\sigma_b)^j}+D(\sigma)y_0(\sigma,t) \tag{3.28}$$

We compute $Y(t)$ using (3.6a), and shifting the contour:

$$\begin{aligned}Y(t)&=\omega^{-1}\int_{0_+}^{\omega_-}e^{-i\sigma t}y(\sigma,t)d\sigma\\&=\omega^{-1}\int_{0_+}^{-iK(M)+0_+}e^{-i\sigma t}y(\sigma,t)d\sigma+\omega^{-1}\int_{-iK(M)}^{-iK(M)+\omega}e^{-i\sigma t}y(\sigma,t)d\sigma\\&+\omega^{-1}\int_{-iK(M)+\omega_-}^{\omega_-}e^{-i\sigma t}y(\sigma,t)d\sigma+\text{Residues}=\omega^{-1}\int_{-iK(M)}^{-iK(M)+\omega}e^{-i\sigma t}y(\sigma,t)d\sigma\\&+\omega^{-1}\int_{0}^{-iK(M)+0}e^{-i\sigma t}y(\sigma+0_+,t)-e^{-(i\sigma+\omega_-)t}y(\sigma+\omega_-,t)d\sigma\\&+\text{Residues}\end{aligned} \tag{3.29}$$

If $[1-K(\sigma)]^{-1}$ has more than M poles, then we make $K(M)$ sufficiently large to collect M of them; otherwise, we simply collect all the poles. The residue term is given by:

$$\sum_{k=0}^{M-1}\sum_{j=0}^{n_k}\alpha_{k,j}t^je^{-i\check{\sigma}_kt}Y_{k,j}(t)$$

stemming from the M poles with $\Im\check{\sigma}_k>-K(M)$. By (3.6c), we can change the integral in the second to last line of (3.29) to:

$$\omega^{-1}\int_0^{-iK(M)}e^{-i\sigma t}(y(\sigma+0_+,t)-y(\sigma+0_-,t))d\sigma \tag{3.30}$$

Note that $y(\sigma,t)$ is analytic in $\sigma^{1/2}$, and thus $y(\sigma+0_+,t)-y(\sigma+0_-,t)$ can be expanded in a Puiseux series in $\sigma^{1/2}$ (and a Fourier series in t). Watson's Lemma yields:

$$\begin{aligned}(3.30)=\omega^{-1}\int_0^{-iK(M)}&e^{-i\sigma t}\sum_{n\in\mathbb{Z}}e^{-in\omega t}\sum_{j=0}^{\infty}D_{j,n}\sigma^{j/2}d\sigma\\&\sim\omega^{-1}\sum_{n\in\mathbb{Z}}e^{in\omega t}\sum_{j=3}^{\infty}D_{j,n}\Gamma(j/2)t^{-j/2}\end{aligned} \tag{3.31}$$

This is what we wanted to show.

When $\check{\sigma}_k=0$, the result follows simply by noting that the sum over j in (3.30) starts from $j=-1$ rather than $j=0$, thereby letting the sum on the right of (3.30) start at $j=1$ instead of $j=3$.

The integral from $-iK(M)$ to $-iK(M)+\omega$ decays at least as fast as $O(e^{-K(M)t})$, and is included in $D_k(t)$. $\square$

We now reconstruct the wavefunction in the magnetic gauge. The basic idea is as follows. We know that $\psi_B(0,t) = D_M(t) + \sum_{k=0}^{M}\sum_{j=0}^{n_k}\alpha_{k,j}t^j e^{-i\tilde{\sigma}_k t}Y_{k,j}(t)$. Using the fact that $\delta(x)\psi_B(x,t) = \delta(x)\psi_B(0,t)$, we find that $\psi_B(x,t)$ satisfies the following equation:

$$\begin{aligned} i\partial_t\psi_B(x,t) &= \left(-\partial_x^2 + 2ib(t)\partial_x\right)\psi_B(x,t) - 2\delta(x-c(t))\psi_B(x,t) \\ &= \left(-\partial_x^2 + 2ib(t)\partial_x\right)\psi_B(x,t) - 2\delta(x-c(t))\psi_B(0,t) \\ &= \left(-\partial_x^2 + 2ib(t)\partial_x\right)\psi_B(x,t) - 2\delta(x-c(t))Y(t) \qquad (3.32) \end{aligned}$$

We will use, for $\Im\sigma < 0$ a solution operator for the Floquet problem $G(\sigma)$ (described shortly) to extend (3.26) to all x, thereby recovering (1.6). The Gamow vectors will come from applying $G(\sigma)$ to $Y_{k,j}(t)$, while the dispersive part will come from applying this operator to $D_M(t)$.

Proposition 3.17 *Let $G(\sigma)$ be the solution operator for the equation:*

$$(\sigma + i\partial_t + \partial_x^2 - b(t)\partial_x)u(x,t) = -2\delta(x)f(t) \qquad (3.33)$$

so that for $\Im\sigma > 0$, $u(x,t)$ decays as $x \to \pm\infty$. By "solution operator", we mean that $G(\sigma)$ maps $f(t) \mapsto u(x,t)$. Then $G(\sigma)$ can be analytically continued to the region $\Im\sigma \leq 0$. The function $u(x,t) = G(\sigma)[-2\delta(x)f(t)]$ has the expansion:

$$u(x,t) = \left\{ \begin{array}{ll} \sum_m u_{m,R}e^{\lambda_{m,-}x}2^{-1/2}\lambda_{m,-}^{-1}e^{-im\omega t}e^{\mp\lambda_{m,-}c(t)}, & x \geq 0 \\ \sum_m u_{m,R}e^{\lambda_{m,+}x}2^{-1/2}\lambda_{m,+}^{-1}e^{-im\omega t}e^{\mp\lambda_{m,+}c(t)}, & x \leq 0 \end{array} \right. \qquad (3.34a)$$

$$\lambda_{m,\pm} = \mp i\sqrt{\sigma + m\omega} \qquad (3.34b)$$

where $f(t) \mapsto \{u_{m,R}, u_{m,L}\}$ is a mapping from $L^2(S^1_\omega) \to l^2(\mathbb{Z}\times\{L,R\})$. $G(\sigma)$ is also a continuous map, analytic in $\sigma^{1/2}$ from $L^2(S^1_\omega) \to L^2(B_R \times S^1_\omega)$ with $B_R = \{x : |x| < R\}$ for any (fixed) R. Near $\sigma = 0$, we have $G(\sigma)\delta(x)f(t) = \sigma^{-1/2}(1/2)P_0 f(t) + O(1)$, with $P_0 f(t)$ the projection onto the zero'th Fourier coefficient of $f(t)$ and the $O(1)$ term being analytic in $\sigma^{1/2}$.

This result is proved in Appendix B. We are now ready to compute the resonance decomposition, (1.6).

Proposition 3.18 *The expansion* (1.6) *holds.*

Proof. We work in the magnetic gauge, to simplify this part of the problem. Note that $\psi_B(x,t) = \psi_v(x+c(t),t)$, so in particular, $\psi_B(0,t) = \psi_v(c(t),t) = Y(t)$. Moreover, recall that the Zak transform commutes with periodic operators, such as the coordinate transform $(x,t) \mapsto (x+c(t),t)$.

Additionally, in what follows, the notation $A(\sigma^{1/2})$ denotes a function analytic in $\sigma^{1/2}$ taking values in $L^2(S^1_\omega, dt)$. Note that $A(\sigma^{1/2})$ may vary from line to line and from equation to equation.

Define $\psi_I(x,t) = e^{i\partial_x^2 t} e^{ic(t)\partial_x}\psi_0(x)$ to be the solution of the Schrödinger equation in the magnetic gauge with no binding potential, and initial condition $\psi_0(x)$. Define $\psi_S(x,t)$ to be the scattered part of $\psi(x,t)$, i.e. the solution of (3.32) with $\psi_S(x,0) = 0$. This implies that $\psi_B(x,t) = \psi_I(x,t) + \psi_S(x,t)$.

By Zak transforming (3.32), we obtain the following equation for $\Psi_S(\sigma,x,t) = \mathcal{Z}[\psi_S](\sigma,x,t)$:

$$(\sigma + i\partial_t)\Psi_S(\sigma,x,t) = [-\Delta + 2ib(t)\partial_x]\, y(\sigma,t)\Psi_S(\sigma,x,t) - 2\delta(x)y(\sigma,t)$$

Equivalently:

$$\Psi_S(\sigma,x,t) = -\left[+\sigma + i\partial_t + \Delta - b(t)\partial_x\right]^{-1} 2\delta(x)y(\sigma,t) \tag{3.35}$$

This has the solution $\Psi_S(\sigma,x,t) = -G(\sigma)2\delta(x)y(\sigma,t)$.

Note that by Proposition 3.6, for each x, $\Psi_I(\sigma,x,t) = \mathcal{Z}[\psi_I](\sigma,x,t)$ takes the form $(1/2)\sigma^{-1/2}\int\psi_0(x)dx + f(\sigma^{1/2},x,t)$ with $f(\sigma^{1/2},x,t)$.

We can now reconstruct $\psi_B(x,t)$ by inverting the Zak transform of $\Psi(\sigma,x,t) = \Psi_I(\sigma,x,t) + \Psi_S(\sigma,x,t)$:

$$\begin{aligned}\psi_B(x,t) &= \omega^{-1}\int_0^{\omega} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma \\ &= \int_0^{-iK(M)} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma + \int_{-iK(M)}^{-iK(M)+\omega} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma + \int_{-iK(M)+\omega}^{\omega} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma \\ &\quad + \text{Residues}\end{aligned} \tag{3.36}$$

Note that both $\Psi_I(\sigma,x,t)$ and $\Psi_S(\sigma,x,t)$ are bounded on for fixed x, and for $\Im\sigma = -iK(M)$. Thus, the integral over the contour $[-iK(M), -iK(m)+\omega]$ decays like $O(e^{-K(M)t})$. Thus, (3.36) becomes:

$$\begin{aligned}&\psi(x,t) \\ &= \int_0^{-iK(M)} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma - \int_{\omega}^{-iK(M)+\omega} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma + \text{Residues} + O(e^{-K(M)t})\end{aligned} \tag{3.37}$$

We show that the contour integral in (3.37) gives rise to the dispersive part, while residues give rise to the resonance.

The Residue Term, $\check{\sigma}_k \neq 0$

By substituting (3.23) into (3.36), we find that when $\check{\sigma}_k \neq 0$, the residue term (for each pole) takes the form:

$$-e^{-i\check{\sigma}_k t}t^j \frac{2}{\omega j!}G(\check{\sigma}_k)\delta(x)Y_{k,j}(t)\langle Y_{k,j}(t)|y_0(\check{\sigma},t)\rangle = \alpha_{j,k}e^{-i\check{\sigma}_k t}t^j\Phi_{k,j}(x,t) \tag{3.38}$$

with $\alpha_{k,j} = \langle Y_{k,j}(t)|y_0(\check{\sigma},t)\rangle$ and $\Phi_{k,j}(x,t) = G(\check{\sigma}_k)\delta(x)Y_{k,j}(t)$. This follows because $y(\sigma,t)$ has a pole at $\sigma = \check{\sigma}_k$ with residue $Y_{k,j}(t)$, and analyticity of $G(\sigma)$ (recall Proposition 3.17, in particular (3.34a)) shows that $\Psi(\sigma,x,t)$ has a pole at $\sigma = \check{\sigma}_k$ with residue $\Phi_{k,j}(x,t)$. Thus we have proved (1.4c).

The Dispersive Part

To compute the integral term of (3.37), note that we must compute:

$$\Psi_M(x,t) = \int_0^{-iK(M)} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma - \int_\omega^{-iK(M)+\omega} e^{-i\sigma t}\Psi(\sigma,x,t)d\sigma =$$

$$\int_0^{-iK(M)} e^{-i\sigma t}\Psi_I(\sigma,x,t)d\sigma - e^{-i(\sigma+\omega)t}\Psi_I(\sigma+\omega,x,t)d\sigma$$

$$-2\int_0^{-iK(M)} e^{-i\sigma t}G(\sigma)\delta(x)y(\sigma,t) - e^{-i(\sigma+\omega)t}G(\sigma+\omega)\delta(x)y(\sigma+\omega,t)d\sigma$$

$$+O(e^{-K(M)t}) \quad (3.39)$$

Since $\mathcal{Z}[f](\sigma+\omega,t) = e^{i\omega t}\mathcal{Z}[f](\sigma,t)$, we find that:

$$e^{-i\sigma t}\left(\Psi_I(\sigma,x,t) - e^{-i\omega t}\psi_I(\sigma+\omega,x,t)\right) = e^{-i\sigma t}(\Psi_I(\sigma,x,t) - \Psi_I(\sigma-0,x,t)) \quad (3.40)$$

Using the fact that $\Psi_I(\sigma,x,t) = \sigma^{-1/2}(1/2)\int\psi_0(x)dx + A(\sigma^{1/2})$ (by Proposition 3.6) we find that:

$$(3.40) = e^{-i\sigma t}(\sigma+0)^{-1/2}(1/2)\int\psi_0(x)dx + A(\sigma^{1/2})$$

$$-e^{-i\sigma t}(\sigma-0)^{-1/2}(1/2)\int\psi_0(x)dx + A(\sigma^{1/2})$$

$$= e^{-i\sigma t}\sigma^{-1/2}\int\psi_0(x)dx + e^{-i\sigma}A(\sigma^{1/2})$$

Plugging this into (3.39) yields:

$$(3.39) = t^{-1/2}\int\psi_0(x)dx + \int_0^{-iK(M)} e^{-i\sigma t}A(\sigma^{1/2})d\sigma$$

$$-2\int_0^{-iK(M)} e^{-i\sigma t}G(\sigma)\delta(x)y(\sigma,t) - e^{-i(\sigma+\omega)t}G(\sigma+\omega)\delta(x)y(\sigma+\omega,t)d\sigma$$

$$+O(e^{-K(M)t}) \quad (3.41)$$

Again using the identity $\mathcal{Z}[y](\sigma+\omega,t) = e^{i\omega t}\mathcal{Z}[y](\sigma,t)$, we find:

$$(3.41) = t^{-1/2}\int\psi_0(x)dx + \int_0^{-iK(M)} e^{-i\sigma t}A(\sigma^{1/2})d\sigma$$

$$-2\int_0^{-iK(M)} e^{-i\sigma t}\left[G(\sigma+0)\delta(x)y(\sigma+0,t) - G(\sigma-0)\delta(x)y(\sigma-0,t)\right]d\sigma$$

$$+O(e^{-K(M)t}) \quad (3.42)$$

Since $G(\sigma)\delta(x) = \sigma^{-1/2}(1/2)P_0 + (C + A(\sigma^{1/2}))$ near $\sigma = 0$, the second to last line of (3.42) becomes:

$$\int_0^{-iK(M)} e^{-i\sigma t}\Big[(\sigma+0)^{-1/2}(1/2)P_0 y(\sigma+0,t) + (C + A(\sigma^{1/2}))y(\sigma+0,t)$$
$$- (\sigma-0)^{-1/2}(1/2)P_0 y(\sigma,t) + (C + A(\sigma^{1/2}))\delta(x)y(\sigma-0,t)\Big]d\sigma$$
$$= \int_0^{-iK(M)} e^{-i\sigma t}\sigma^{-1/2}P_0 y(0,t) + e^{-i\sigma t}A(\sigma^{1/2})d\sigma \quad (3.43)$$

If $(1-K(\sigma))^{-1}$ has no poles near $\sigma = 0$, then $P_0 y(0,t) = (1/2)\int \psi_0(x)dx$ (see Proposition 3.12), and plugging (3.43) into (3.41) yields:

$$\Psi_M(x,t) = t^{-1/2}\int \psi_0(x)dx + \int_0^{-iK(M)} e^{-i\sigma t}A(\sigma^{1/2})d\sigma$$
$$- \int_0^{-iK(M)} e^{-i\sigma t}\sigma^{-1/2}\left[\int \psi_0(x)dx\right]d\sigma + \int_0^{-iK(M)} e^{-i\sigma t}A(\sigma^{1/2})d\sigma$$
$$= \int_0^{-iK(M)} e^{-i\sigma t}A(\sigma^{1/2})d\sigma + O(e^{-K(M)t}) \quad (3.44)$$

The integral in the last line of (3.44) is a Laplace-type integral, and $A(\sigma^{1/2})$ is analytic in $\sigma^{1/2}$. Watson's lemma therefore yields (1.7), and analyticity in $\sigma^{1/2}$ shows that the sum in (1.7) starts at $n = 3$. If $(1-K(\sigma))^{-1}$ has poles near $\sigma = 0$ (i.e. $\check{\sigma}_k = 0$ for some k), then the sum will begin from $n = 1$ (see below).

The Residue Term, $\check{\sigma}_k = 0$

In the event that $(1-K(\sigma))^{-1}$ has poles in $\sigma^{1/2}$ near $\sigma = 0$, the only difference in the above analysis is that the $\sigma^{-1/2}$ terms coming from $y_0(\sigma,t)$ will not cancel the $\sigma^{-1/2}$ terms coming from $G(\sigma)\delta(x)y(\sigma,t)$. Thus, (3.44) instead becomes:

$$\Psi_M(x,t) = \int_0^{-iK(M)} e^{-i\sigma t}\left[\sum_{n=1}^{M'} \frac{d_j(x,t)}{\sigma^{-n/2}} + A(\sigma^{1/2})\right]d\sigma + O(e^{-iK(M)t})$$

The order of the pole, M', can not be larger than 2 since this would imply that $\Psi_M(x,t)$ grows at the rate $t^{M'/2-1}$, which would contradict conservation of probability. If the order of the pole is 2, this corresponds to a Floquet bound state at zero energy, and if the order is 1, this corresponds to a zero energy resonance, and (1.6) holds with the sum starting from $n = 1$ in (1.7). □

We have thus far proved all of Theorem 2 except for the fact that $\Phi_{k,0}(x,t)$ decays at $x = \pm\infty$ if $\Im\check{\sigma}_k = 0$.

Proposition 3.19 *Suppose that $\Im\check{\sigma}_k = 0$. Then $\Phi_{k,0}(x,t)$ decays at $x = \pm\infty$ and $\psi_n^{L,R} = 0$ for all $n < 0$. Furthermore, the pole is of order 1.*

Proof. It is clear that unitary evolution implies:

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-R}^{R}|\psi(x,t)|^2\,dxdt \leq 1 \tag{3.45}$$

If the pole is of order greater than 1, then:

$$\psi(x,t) = \sum_{j=0}^{n_k} t^j \alpha_{k,j} e^{-i\check{\sigma}_k t}\Phi_{k,j}(x,t) + \sum_{k'\neq k} t^j \alpha_{k',j} e^{-i\check{\sigma}_{k'} t}\Phi_{k',j}(x,t) + \Psi_M(x,t)$$

But the second two terms decay, while the first grows with time. This contradicts unitary evolution, unless $n_k = 0$. Thus the pole must be of first order.

Now suppose that in the expansion of $\Phi_{k,0}(x,t)$, at least one $\psi_n^{L,R} \neq 0$ with $n < 0$. Then $\Phi_{k,0}(x,t)$ will oscillate with x rather than decay. This implies that:

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-R}^{R}|\Phi_{k,j}(x,t)|^2\,dxdt \geq CR$$

for sufficiently large R and some $C > 0$. On the other hand, the rest of $\psi(x,t)$ (the dispersive part, and the exponentially decaying poles) which we denote $R(x,t)$ decays with time. This implies that for $t \geq t_R$ (with T_R chosen large enough so that $|R(x,t)| \leq \epsilon/\sqrt{2R}$ that:

$$\|\psi(x,t)\| \;=\; \|\Phi_{k,0}(x,t) + R(x,t)\| \;\geq\; \|\Phi_{k,0}(x,t)\| - \|R(x,t)\| \;\geq\; \sqrt{CR} - \epsilon$$

Selecting $R > (2+\epsilon)/C$ causes $\|\psi(x,t)\| \geq 1$, contradicting unitary evolution.

Intuitively, what this means is the following. The modes $\psi_n^{L,R}$ with $n < 0$ correspond to radiation modes. If such a mode is nonzero, then $\Phi_{k,0}(x,t)$ will be emitting "radiation" without decaying, which is clearly impossible. □

4 Concluding Remarks

In this paper we studied the interaction of a simple model atom with a dipole radiation field of arbitrary strength. We obtained a resonance expansion, in which resonances can be resolved regardless of their complex quasi-energy. In particular, we obtained a rigorous definition of the ionization rate $\gamma = -2\Im\check{\sigma}_k$ and Stark-shifted energy, $\Re\check{\sigma}_k$ for the k-th resonance. We further showed that complete ionization occurs ($\gamma > 0$) when $E(t)$ is a trigonometric polynomial.

Some possible future directions of research include:

4.1 Perturbative and numerical calculations

The main feature of our method is that it turns a time dependent problem on $\mathbb{R}$ into a compact analytic Fredholm integral equation. This implies that a family of finite dimensional approximations can be used (in the Zak domain) to approximate solutions to the time dependent Schrödinger equation.

We believe that the quasi-energy methodology used here and in related papers [6, 9, 8] can be used for quantitative calculations of realistic physical systems. Perturbative calculations along these lines have recovered Fermi's Golden Rule and the multiphoton effect.

4.2 Extension to 3 dimensions

In the case of $H_0 = -\Delta - 2\delta(\vec{x})$ with $\vec{x} \in \mathbb{R}^3$, a similar equation to (3.4) can be derived. Due to the fact that $\delta(\vec{x})$ is not in $H^{-1}(\mathbb{R}^3)$, $\psi(\vec{x}, t)$ becomes singular at $t = 0^+$. This can be remedied by considering weak solutions[7] [30], and an equation similar in most respects to (3.4) can be derived which governs the evolution [14]. For this reason, we believe most results can be adapted to this case, as has been done for $H_0 = -\Delta - 2\delta(x) + E(t)\delta(x)$ [9, 6, 26].

A Proof of Proposition 2.2

We observe that by the results of Section 3, if a bound state exists, then:

$$\psi_B(0, t) = e^{a(t)/4} e^{-ia(t)} Y_k(t)$$

Setting $z = e^{-i\omega t}$, and $y(z) = Y_k(t)$, we wish to show that $y(z) = f(z) + g(z)$ with f, g both entire of exponential order $2n$. This is equivalent to showing that:

$$|Y_k(t + i\alpha)| \leq C \exp[C' \exp(|2N\omega\alpha|)]$$

The function $Y_k(t)$ satisfies the equation:

$$Y_k(t) = \int_0^{2\pi/\omega} k'(t, s) Y_k(t - s) ds = -\int_0^{2\pi/\omega} k'(t, t - s) Y_k(s) ds$$

with $k'(t, s)$ as defined in (3.19b). Thus we obtain the bound:

$$|Y_k(t + i\alpha)| \leq \int_0^{2\pi/\omega} |k'(t + i\alpha, t + i\alpha - s)| \, |Y_k(s)| \, ds \tag{A.1}$$

and it suffices to bound $|k'(t + i\alpha, t + i\alpha - s)|$. From the definition of $k'(t, s)$, we find:

$$\begin{aligned} &k'(t + i\alpha, t + i\alpha - s) \\ &\quad = \frac{\omega}{2\pi i} \int_{\mathbb{R}+0i} \frac{e^{\sigma p}}{1 - e^{\omega p + \alpha - i\omega(t-s)}} \left[\exp\left(\frac{(c(t + i\alpha) - c(s))^2}{4p} \right) - 1 \right] \frac{dp}{\sqrt{p}} \end{aligned}$$

Supposing $\alpha/\omega > 1$ (we are interested in the behavior as $\alpha \to \infty$), the integrand is analytic for $z = re^{i\theta}$, $0 < r < 1$ and $0 \leq \theta \leq \pi$. Thus, we can deform the contour from $\mathbb{R} + 0i$ to $\gamma = \partial\{z : \Im z < 0 \,\text{or}\, |z| < 1\}$.

[7] One considers the operator H_0 restricted to the domain $D = \{f(\vec{x}) : f(x) \in H^1(\mathbb{R}^3)$ and $f(0) = 0\}$. From this domain, one can construct a self-adjoint extension of H_0, thus allowing the evolution to be defined.

Note that for some constant C, $|c(t+i\alpha)| \leq Ce^{N\omega|\alpha|}$, since $c(t)$ is a trigonometric polynomial of order N.

We find that there are three regions of integration which contribute to $k'(t+i\alpha, t+i\alpha-s)$. The regions of integration contributing come from the region near $1-e^{\omega p+\alpha-i\omega(t-s)}=0$ (the pole of the integrand), large p and small p.

If the pole is closer to $\mathbb{R}$ than π/ω, we deform γ up to encircle it at a distance $pi\omega$. Otherwise, we ignore it. Therefore, in any case, for $z\in\gamma$, $1-e^{\omega p+\alpha-i\omega(t-s)}$ is uniformly bounded away from zero.

We then split $\gamma=\gamma_<\cup\gamma_>\cup\gamma_\alpha$ where $\gamma_<=\{p\in\gamma:|p|<(Ce^{N\omega|\alpha|}+\|c(s)\|_{L^\infty})^2\}$ and $\gamma_>=\gamma\setminus\gamma_<$. We therefore find that:

$$\begin{aligned}
&|k'(t+i\alpha,t+i\alpha-s)| \leq |\text{residue}| \\
&\quad C\int_{\gamma_<}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}\left[\exp\left(\frac{(c(t+i\alpha)-c(s))^2}{4p}\right)-1\right]\right|\frac{dp}{\sqrt{|p|}} \\
&\quad +C\int_{\gamma_>}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}\left[\exp\left(\frac{(c(t+i\alpha)-c(s))^2}{4p}\right)-1\right]\right|\frac{dp}{\sqrt{|p|}} \\
&\qquad \leq C
\end{aligned}$$

The residue can be bounded by:

$$\begin{aligned}
&|\text{residue}| \\
&\leq C\left|e^{\sigma(-\alpha+i\omega(t-s))/\omega}\left[\exp\left(\frac{(c(t+i\alpha)-c(s))^2}{4(-\alpha+i\omega(t-s))/\omega}\right)-1\right]\frac{1}{\sqrt{(-\alpha+i\omega(t-s))/\omega}}\right| \\
&\qquad \leq C\exp(C|c(t+i\alpha)|^2)\leq C\exp(C\exp(2N\omega|\alpha|))
\end{aligned}$$

We bound the integral over the compact region $\gamma_<$ simply by taking absolute values:

$$\begin{aligned}
&\int_{\gamma_<}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}\left[\exp\left(\frac{(c(t+i\alpha)-c(s))^2}{4p}\right)-1\right]\right|\frac{dp}{\sqrt{|p|}} \\
&\qquad \leq |\gamma_<|\,C\exp(C\exp(2N\omega|\alpha|))
\end{aligned}$$

For the integral over $\gamma_>$, we use the fact that if $|z|<1$, $|e^z-1|\leq e|z|$:

$$\begin{aligned}
&\int_{\gamma_>}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}\left[\exp\left(\frac{(c(t+i\alpha)-c(s))^2}{4p}\right)-1\right]\right|\frac{dp}{\sqrt{|p|}} \\
&\qquad \int_{\gamma_>}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}\frac{(Ce^{N\omega|\alpha|}+\|c(s)\|_{L^\infty})^2}{|p|}\right|\frac{dp}{\sqrt{|p|}} \\
&\quad \leq Ce^{2N\omega|\alpha|}\int_{\gamma_>}\left|\frac{e^{\sigma p}}{1-e^{\omega p+\alpha-i\omega(t-s)}}p^{-3/2}\right|dp\leq C\exp(C\exp(2N\omega|\alpha|))
\end{aligned}$$

Combining these estimates, we find that $k'(t+i\alpha,t+i\alpha-s)$ has the required growth as $\alpha\to\infty$, hence $Y_k(t)$ does. The same argument applies as $\alpha\to-\infty$.

B Proof of Proposition 3.17

We state a few results we need.

Theorem 4 (T. Kato, [23, page 368]) *If a family $T(\sigma)$ of closed operators on X depending on σ holomorphically has a spectrum consisting of two separated parts, the subspaces of X corresponding to the separated parts also depend on σ holomorphically.*

Remark B.1 A few words of explanation are in order. In [23], they are given in the commentary following the theorem.

The analytic dependence of the separated parts of the spectrum means the following. Let M_σ, M'_σ be the spectral subspaces of $T(\sigma)$, related to the two separated parts. Then there exists an analytic function $U(\sigma)$ (called the transformation function), with analytic inverse, so that $M_\sigma = U(\sigma)M_0$ and $M'_\sigma = U(\sigma)M'_0$. For fixed σ, both $U(\sigma)$ and $U^{-1}(\sigma)$ are bounded operators on the Hilbert space.

In addition, the spectral projections $P_M(\sigma)$ and $P_{M'}(\sigma)$ can be written as:

$$P_M(\sigma) = U(\sigma)P_M(0)U^{-1}(\sigma) \tag{B.1a}$$
$$P_M(\sigma) = U(\sigma)P_{M'}(0)U^{-1}(\sigma) \tag{B.1b}$$

We now prove a Lemma which allows us to reconstruct $\Psi(\sigma, x, t)$ given solely information about $\Psi(\sigma, 0, t)$. The basic idea is to treat the Schrödinger equation as an evolution equation in x, with a "Hamiltonian" that is periodic in t.

Lemma B.2 *Define the Hilbert space $\mathcal{H} = H^{1/2}(S^1_\omega, dt) \oplus L^2(S^1_\omega, dt)$. Then there exists a sequence N_m with $0 < \inf_m |N_m| \leq \sup_m |N_m|$ so that*

$$\phi_{m,\pm} \stackrel{\text{def}}{=} N_m \begin{pmatrix} 2^{-1/2}\lambda_{m,\pm}^{-1} e^{-im\omega t} e^{\mp\lambda_{m,\pm}c(t)} \\ 2^{-1/2} e^{-im\omega t} e^{\mp\lambda_{m,\pm}c(t)} \end{pmatrix} \tag{B.2a}$$

is a Riesz basis for $\mathcal{H}$. Here, $\lambda_{m,\pm}$ is defined as:

$$\lambda_{m,\pm} \stackrel{\text{def}}{=} \mp i\sqrt{\sigma + m\omega} \tag{B.2b}$$

Furthermore, the operator

$$H \stackrel{\text{def}}{=} \left[\begin{pmatrix} 0 & 1 \\ \sigma + i\partial_t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b(t) \end{pmatrix} \right]$$

is diagonal in this basis, with $H\phi_{m,\pm} = \lambda_{m,\pm}\phi_{m,\pm}$.

Moreover, if we define $\mathcal{H}^+$ as the span of $\{\phi_{m,+}\}_{m\in\mathbb{Z}}$ and $\mathcal{H}^-$ as the span of $\{\phi_{m,-}\}_{m\in\mathbb{Z}}$ then e^{xH} is defined, bounded and analytic in σ for $\Re\sigma \in [0, \omega)$ (except at $\sigma = 0$) on $\mathcal{H}^+$ for $x \leq 0$, and on $\mathcal{H}^-$ for $x \geq 0$.

We are nearly ready to prove Lemma B.2. First a minor technical point.

Remark B.3 Consider the sequence $\sqrt{\sigma + n\omega}$, with $\Re\sigma \in (0, \omega)$. For n negative, $\Im\sqrt{\sigma + n\omega}$ grows like $\sqrt{|n|}$. For n positive, $\Im\sqrt{\sigma + n\omega} = O(n^{-1/2})$, and is uniformly bounded below.

Proof of Lemma B.2. It is a simple calculation to show $\phi_{m,\pm}$ are eigenvectors of H with eigenvalues $\lambda_{m,\pm}$. To show that $\{\phi_m^{\pm}\}$ is a Riesz basis for $\mathcal{H}$, we show that H is a bounded perturbation of a normal operator. Consider the family of operators (analytic in ζ) on $\mathcal{H}$:

$$H_\zeta \begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sigma + i\partial_t & 0 \end{pmatrix} \begin{pmatrix} u \\ u_x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \zeta b(t) \end{pmatrix} \begin{pmatrix} u \\ u_x \end{pmatrix}$$

Consider also the family of vectors (parameterized by ζ):

$$\{\phi_{m,\zeta}^{\pm}\} \quad = \quad \left\{ N_{m,\zeta} \begin{pmatrix} 2^{-1/2}\lambda_{m,\pm}^{-1} e^{-im\omega t} e^{\zeta\lambda_{m,\pm}c(t)} \\ 2^{-1/2} e^{-im\omega t} e^{\mp\zeta\lambda_{m,\pm}c(t)} \end{pmatrix} \right\}_{m\in\mathbb{Z}}$$

$N_{m,\zeta}$ is a normalizing constant which is defined implicitly; we discuss it below. For $\zeta = 0$, $N_{m,\zeta} = 1$.

A simple calculation shows that $(\phi_{m,\zeta}^{\pm}, \lambda_{m,\pm})$ are eigenvector/eigenvalue pairs of H_ζ. In particular, each $\lambda_{m,\pm}$ is separate from all the others. For $\zeta = 0$, they are also orthonormal in $\mathcal{H}$. Let $P_m^{\pm}(\zeta)$ be the associated spectral projection operators, given by

$$P_m^{\pm}(\zeta) = \int_{\gamma_m} (H_\zeta - z)^{-1} dz$$

where γ_m is a closed curve containing only $\lambda_{m,\zeta}^{\pm}$, and no other eigenvalue of H_ζ.

Let $U(\zeta)$ be the transformation function of Theorem 4 (see also Remark B.1 and Eq. (B.1)). Since each $\lambda_{m,\pm}$ is separated from all the others and varies analytically (except near $\sigma = 0$), Theorem 4 (using $\lambda_{m,\pm}$ as one of the separated parts of the spectrum and $\{\lambda_{m',\pm}\}_{m'\neq m}$ as the other) implies that:

$$\begin{aligned} P_m^{\pm}(\zeta) = U(\zeta)P_m^{\pm}(0)U^{-1}(\zeta) = \langle U(\zeta)^{-1} \cdot |\phi_{m,0}^{\pm}\rangle U(\zeta)\phi_{m,0}^{\pm} \\ = \langle \cdot \,|[U(\zeta)^{-1}]^* \phi_{m,0}^{\pm}\rangle U(\zeta)\phi_{m,0}^{\pm} \end{aligned}$$

We know that $U(\zeta)\phi_{m,0}^{\pm}$ is a vector in the direction

$$\begin{pmatrix} 2^{-1/2}\lambda_{m,\pm}^{-1} e^{-im\omega t} e^{\mp\zeta\lambda_{m,\pm}c(t)} \\ 2^{-1/2} e^{-im\omega t} e^{\mp\zeta\lambda_{m,\pm}c(t)} \end{pmatrix}$$

but this determines $U(\zeta)\phi_{m,0}^{\pm}$ only up to a constant (not necessarily real), denoted by $N_{m,\zeta}$. Since $U(\zeta)$ is bounded above and below, $0 < \left\|U(\zeta)^{-1}\right\|^{-1} \leq |N_{m,\zeta}| \leq \|U(\zeta)\|$. To compute the expansion of a function $\psi(t)$ in this basis, we use the formula $\psi_m^{\pm} = \langle U(1)^{-1}\psi|\phi_{m,0}^{\pm}\rangle$. Since $\phi_{m,0}^{\pm}$ is an orthonormal basis, this set of coefficients is clearly in l^2, with l^2 norm bounded below by $\|U(1)\|^{-1} \|\psi(0,t)\|_{\mathcal{H}}$ and above by $\|U(1)\| \|\psi(0,t)\|_{\mathcal{H}}$.

Finally, we need to show that $\sum_m P_m^+(\zeta) + P_m^-(\zeta) = 1$, interpreting the sum in the strong topology. The sum is strongly convergent when $\zeta = 0$, since the $P_m^\pm(0)$ are orthogonal projections. Multiplying on the left and right by the continuous operators $U(\zeta)$ and $U(\zeta)^{-1}$ yields:

$$
\begin{aligned}
1 = U(\zeta)U(\zeta)^{-1} &= U(\zeta)\left(\sum_m P_m^+(0) + P_m^-(0)\right)U(\zeta)^{-1} \\
&= \sum_m U(\zeta)[P_m^+(0) + P_m^-(0)]U(\zeta)^{-1} = \sum_m P_m^+(\zeta) + P_m^-(\zeta)
\end{aligned}
$$

This proves the Riesz basis property. To show boundedness of e^{xH}, simply note that the real part of the eigenvalues of H is bounded above on $\mathcal{H}^+$ and bounded below on $\mathcal{H}^-$ for σ in compact regions not containing $\sigma = 0$. Thus, e^{xH} is bounded on $\mathcal{H}^+$. Analyticity follows by observing that the eigenvalues and eigenfunctions are analytic in $\sigma^{1/2}$, except near $\sigma = 0$. □

We are now prepared to prove Proposition 3.17.

Proof of Proposition 3.17. Note that (3.33) can be rewritten as:

$$
\partial_x \begin{pmatrix} u \\ u_x \end{pmatrix} = H \begin{pmatrix} u \\ u_x \end{pmatrix}
$$

Away from $x = 0$, the solution $u(x,t)$ can be written (formally) as:

$$
\begin{pmatrix} u(x,t) \\ \partial_x u(x,t) \end{pmatrix} = e^{xH} \begin{pmatrix} u(0^\pm,t) \\ \partial_x u(0^\pm,t) \end{pmatrix}, \pm x < 0 \tag{B.3}
$$

At $x = 0$, the two matching conditions need be satisfied:

$$
\begin{aligned}
u(0^+,t) - u(0^-,t) &= \quad 0 \qquad \text{(Continuity)} \\
\partial_x u(0^+,t) - \partial_x u(0^-,t) &= -2f(t) \qquad \text{(Differentiability)}
\end{aligned}
$$

For $\Im\sigma > 0$, $\lambda_{m,+}$ always has positive real part and $\lambda_{m,-}$ always has negative real part (recall (B.2b)). Thus, if $u(x,t)$ is to vanish as $x \to \pm\infty$, we find that:

$$
\begin{aligned}
\begin{pmatrix} u(0^-,t) \\ \partial_x u(0^-,t) \end{pmatrix} &= \sum_m u_{m,R}\phi_{m,-}(t) \\
\begin{pmatrix} u(0^+,t) \\ \partial_x u(0^+,t) \end{pmatrix} &= \sum_m u_{m,L}\phi_{m,+}(t)
\end{aligned}
$$

Since $\phi_{m,\pm}$ is a Riesz basis and $[0, f(t)] \in \mathcal{H}$, we can write:

$$
\begin{pmatrix} 0 \\ f(t) \end{pmatrix} = \sum_m f_{m,+}\phi_{m,+} + f_{m,-}\phi_{m,-} \tag{B.4}
$$

Choosing $u_{m,R} = f_{m,-}$ and $u_{m,L} = -f_{m,+}$ solves (3.33), at least on a formal level. Since $u(0^+,t) \in \mathcal{H}^+$ and $u(0^-,t) \in \mathcal{H}^-$, (B.3) makes sense. Since e^{xH} is bounded and analytic provided $|x| < R$, this is thus an analytic mapping from $L^2(S^1_\omega) \to L^2(B_R \times S^1_\omega)$.

Now observe that both (B.3) and (B.4) can be analytically continued in σ, and the continuation also solves (3.33), therefore $G(\sigma)$ can be analytically continued in σ as well.

We now need only determine the behavior near $\sigma = 0$. By Taylor-expanding (B.2a) in $\sigma^{1/2}$, we find that:

$$\phi_{0,\pm} = N_0 2^{-1/2} \begin{pmatrix} \mp\sigma^{-1/2} \\ 1 \end{pmatrix} + \begin{pmatrix} O(1) \\ O(\sigma^{1/2}) \end{pmatrix} \tag{B.5}$$

while

$$\phi_{m,\pm} = N_m 2^{-1/2} \begin{pmatrix} \lambda_{m,\pm}^{-1} e^{-im\omega t} \\ e^{-im\omega t} \end{pmatrix} + \begin{pmatrix} O(1) \\ O(\sigma^{1/2}) \end{pmatrix}$$

Thus, near $\sigma = 0$, we find to leading order (plugging (B.5) into (B.4)) that:

$$\begin{aligned} f_{m,+} - f_{m,-} &= 0 \\ N_0 2^{-1/2}(f_{m,+} + f_{m,-}) &= P_0 f(t) \end{aligned}$$

with $P_0 f(t)$ projection onto the zero'th Fourier coefficient. This implies that $f_{0,\pm} = u_{0,R} = -u_{0,L} = 2^{1/2} N_0^{-1} (1/2) P_0 f(t) + O(\sigma^{1/2})$. On all other coefficients, the behavior is analytic in σ since $\lambda_{m,\pm}$ is analytic in σ for $m \neq 0$. Thus for small σ:

$$u(x,t) = (1/2)(P_0 f(t)) \begin{pmatrix} \sigma^{-1/2} \\ 1 \end{pmatrix} + O(1)$$

and therefore $G(\sigma)\delta(x)f(t) = (1/2)\sigma^{-1/2}[P_0 f(t)] + O(1)$ near $\sigma = 0$. □

C Wellposedness

Well posedness of (1.10) (and by extension (1.9) and (1.1)) is sketched in an example at the end of [33], though we sketch a proof for completeness. Let H_0 be a self adjoint operator and let $V(t)$ be a time-dependent quadratic form. Suppose the following conditions hold:

$$D(H_0) \subseteq D(V(t)) \tag{C.1a}$$

There exist constants $a \in (0,1)$, $b \in (0,\infty)$ such that:

$$|V(t)(f,f)| \leq a\langle H_0^{1/2} f | H_0^{1/2} f\rangle + b\langle f|f\rangle \tag{C.1b}$$

The function $(H_0+1)^{-1/2} V(t) (H_0+1)^{-1/2}$ is norm differentiable with derivative $(H_0+1)^{-1/2} V'(t)(H_0+1)^{-1/2}$ and

$$|V'(t)(f,f)| \leq a\langle H_0^{1/2} f | H_0^{1/2} f\rangle + b\langle f|f\rangle \tag{C.1c}$$

If $H_0, V(t)$ satisfy these conditions, then by [33, Theorem 11], there is a constant C so that $H_0 + V(t) + C$ is a K-generator (defined in [33]). It is then shown in [33, Theorem 8] that a K-generator generates a unitary propagator $U(t,s)$, i.e.

$$\partial_t \langle f | U(t,s) \psi_0 \rangle = \langle (H_0 + V(t) + C)^{1/2} f | (H_0 + V(t) + C)^{1/2} U(t,s) \psi_0 \rangle. \quad \text{(C.2)}$$

We let $H_0 = -\partial_x^2$ and $V(t) = ib(t)\partial_x - 2\delta(x)$. (C.1a) clearly holds; $D(ib(t)\partial_x) = H^{1/2} \supseteq H^1$, while Sobolev embedding shows that $D(-2\delta(x)) \supseteq H^1$. (C.1c) follows simply by noting that $V'(t) = E(t)\partial_x$, which is again $-\partial_x^2$-bounded for sufficiently large b.

(C.1b) can be verified with $a = 1/2$. Clearly, $ib(t)\partial_x$ is $-\partial_x^2$-bounded with $a = 1/4$ (or any other a). Note that $f(0) = \int \hat{f}(k) dk$. Thus:

$$\begin{aligned} \langle f | 2\delta(x) f \rangle = 2 \left| f(0) \right|^2 = 2 \left| \int \hat{f}(k) dk \right|^2 &= 2 \left| \int \frac{(b + k^2/4)^{1/2}}{(b + k^2/4)^{1/2}} \hat{f}(k) dk \right|^2 \\ &\leq 2 \left\| (b + k^2/4)^{1/2} \hat{f}(k) \right\|_{L^2}^2 \left\| (b + k^2/4)^{-1/2} \right\|_{L^2}^2 \\ &= 2 \left\| (b + k^2/4)^{-1/2} \right\|_{L^2}^2 \left[(1/4) \langle H_0^{1/2} f | H_0^{1/2} f \rangle + b \langle f | f \rangle \right] \end{aligned}$$

We can make $\left\| (b + k^2/4)^{-1/2} \right\|_{L^2}^2 \leq 1/2$ by choosing b large; thus $-2\delta(x)$ is $-\partial_x^2$ bounded with $a = 1/4$. Adding the results together verifies (C.1b).

Thus, (1.10) is well posed. Applying the unitary transformations described in Section 1.2 shows that (1.1) and (1.9) are well posed as well.

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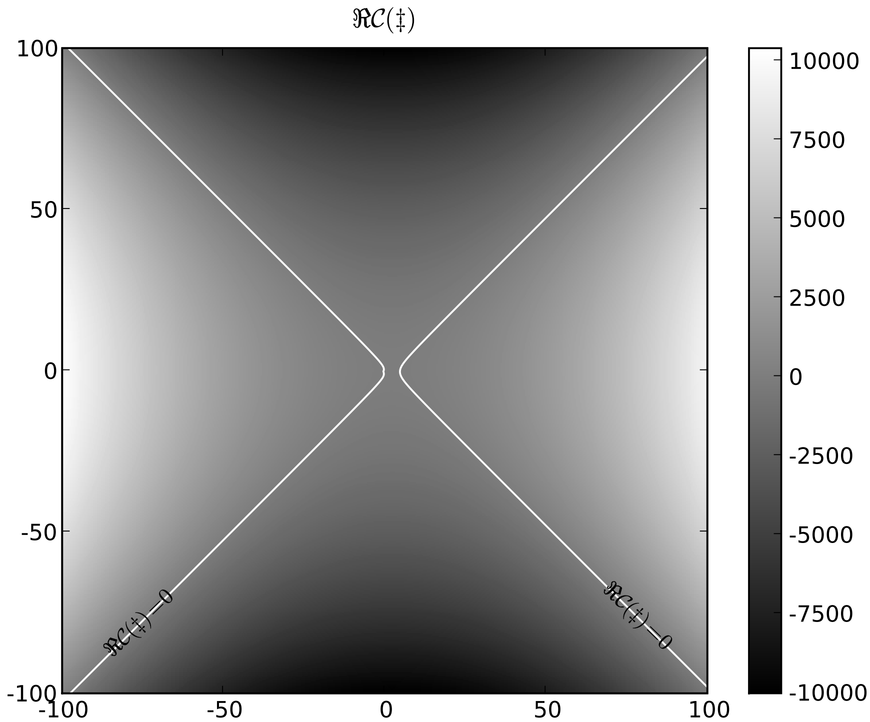

Figure 1: A plot of $\Re\mathfrak{C}(z)$, for the specific choice of $\mathfrak{C}(z) = z^2 - 5z - 5z^{-1} + z^{-2}$. As is apparent from the figure, the regions $\Re\mathfrak{C}(z) > 0$ and $\Re\mathfrak{C}(z) < 0$ are approximately sectors for sufficiently large $|z|$.

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