Stable Outgoing Wave Filters for Anisotropic Waves

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Joint work with Avy Soffer.
Supported by NSF and Bevier Fellowship.

Linear Waves

- Linear wave equation:

$$
\begin{aligned}
\vec{u}_t(x,t) &= H\vec{u}(,t) \\
H(i\nabla) &= -H^\dagger(i\nabla)
\end{aligned}
$$

Linear Waves

- Linear wave equation:

$$\begin{aligned}\vec{u}_t(x,t) &= H\vec{u}(,t) \\ H(i\nabla) &= -H^\dagger(i\nabla)\end{aligned}$$

- Schrodinger equation:

$$\begin{aligned}H &= i\Delta \\ u(x,t) &= \psi(x,t)\end{aligned}$$

Linear Waves

- Linear wave equation:

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\begin{aligned}
\vec{u}_t(x,t) &= H\vec{u}(,t) \\
H(i\nabla) &= -H^\dagger(i\nabla)
\end{aligned}
$$

- Maxwell's equation

$$
\begin{aligned}
H &= \begin{bmatrix} 0 & -\mu^{-1/2}\nabla \times \epsilon^{-1/2} \\ \epsilon^{-1/2}\nabla \times \mu^{-1/2} & 0 \end{bmatrix} \\
\vec{u}(x,t) &= (\sqrt{\mu}\vec{H}, \sqrt{\epsilon}\vec{E})
\end{aligned}
$$

Linear Waves

- Linear wave equation:

$$\begin{aligned} \vec{u}_t(x,t) &= H\vec{u}(,t) \\ H(i\nabla) &= -H^\dagger(i\nabla) \end{aligned}$$

- Linearized Euler equation:

$$\begin{aligned} H &= \begin{bmatrix} M\partial_{x_1} & -\partial_{x_1} & -\partial_{x_2} \\ -\partial_{x_1} & M\partial_{x_1} & 0 \\ -\partial_{x_2} & 0 & M\partial_{x_1} \end{bmatrix} \\ (x,y) &= (p(x,t), v_x(x,t), v_y(x,t)) \end{aligned}$$

Linear Waves

- Linear wave equation:

$$\begin{aligned} \vec{u}_t(x,t) &= H\vec{u}(,t) \\ H(i\nabla) &= -H^\dagger(i\nabla) \end{aligned}$$

- Relativistic Schrodinger Equation

$$\begin{aligned} H &= \sqrt{-\Delta + m^2} - m \\ u(x,t) &= \psi(x,t) \end{aligned}$$

Linear Waves

- Linear wave equation:

$$\begin{aligned} \vec{u}_t(x,t) &= H\vec{u}(,t) \\ H(i\nabla) &= -H^\dagger(i\nabla) \end{aligned}$$

- Linear part of Benjamin-Ono equation:

$$\begin{aligned} H &= |\partial_x|\partial_x \\ u(x,t) &= h(x,t) \end{aligned}$$

Numerical Solution

- Finite Differences
- Finite Elements
- Spectral methods

I'll stay agnostic

FFT spectral methods rock.

Mention Fundamental Complexity

Numerical Solution

- Sample spacing:

$$\delta x \leq O(2\pi / k_{max})$$

- Fundamental complexity of timestepping on $[-L, L]^N$

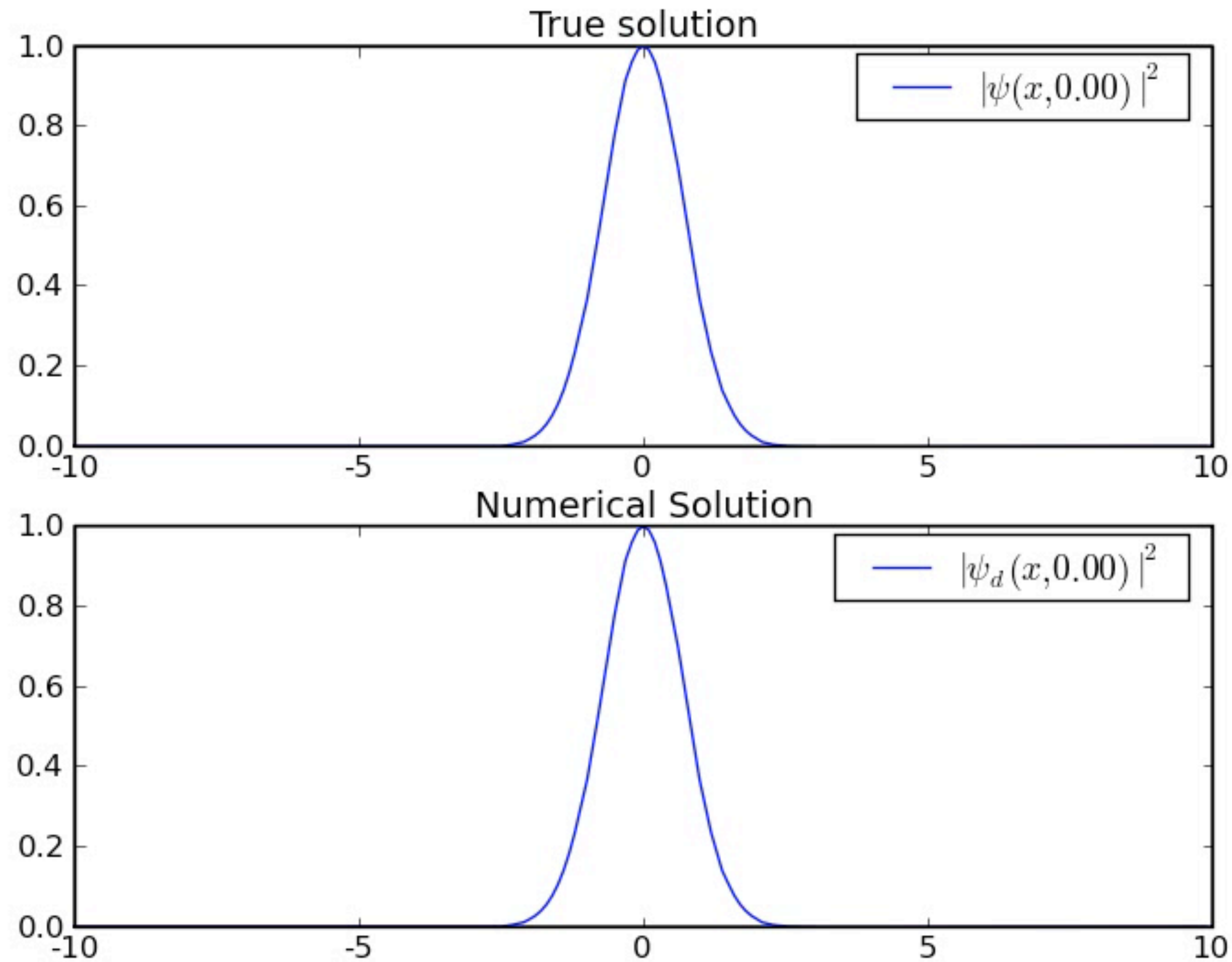
$$\begin{aligned} \text{Memory} &= O((Lk_{max})^N) \\ \text{Complexity} &= O((T_{max}/\delta t)(Lk_{max})^N) \end{aligned}$$

- Solution on $\mathbb{R}^N$ requires careful choice of boundary conditions.

Outgoing Waves are a Problem

The Problem

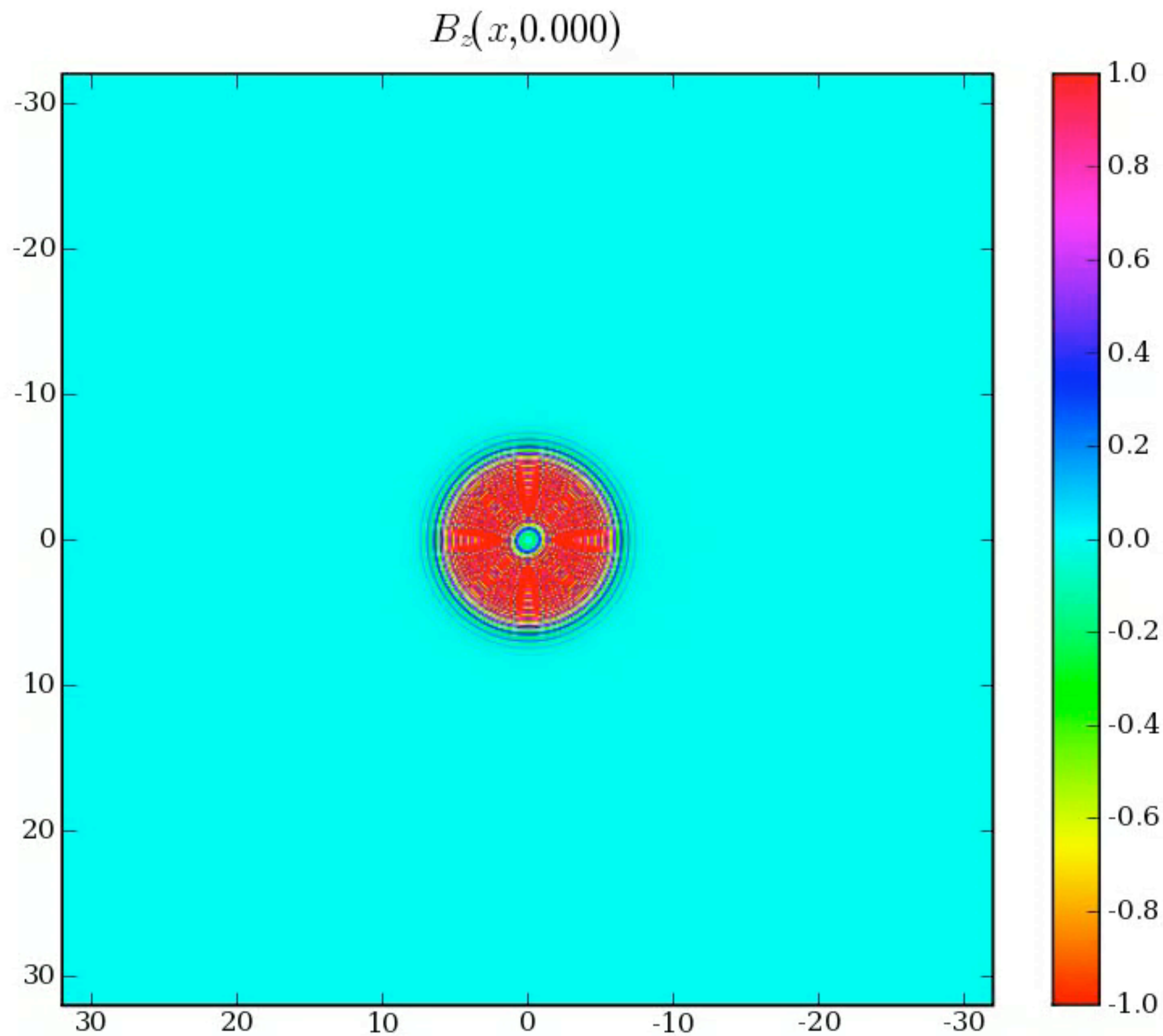
1D Schrodinger Equation



The Problem

Anisotropic Maxwell

Incorrect Boundaries



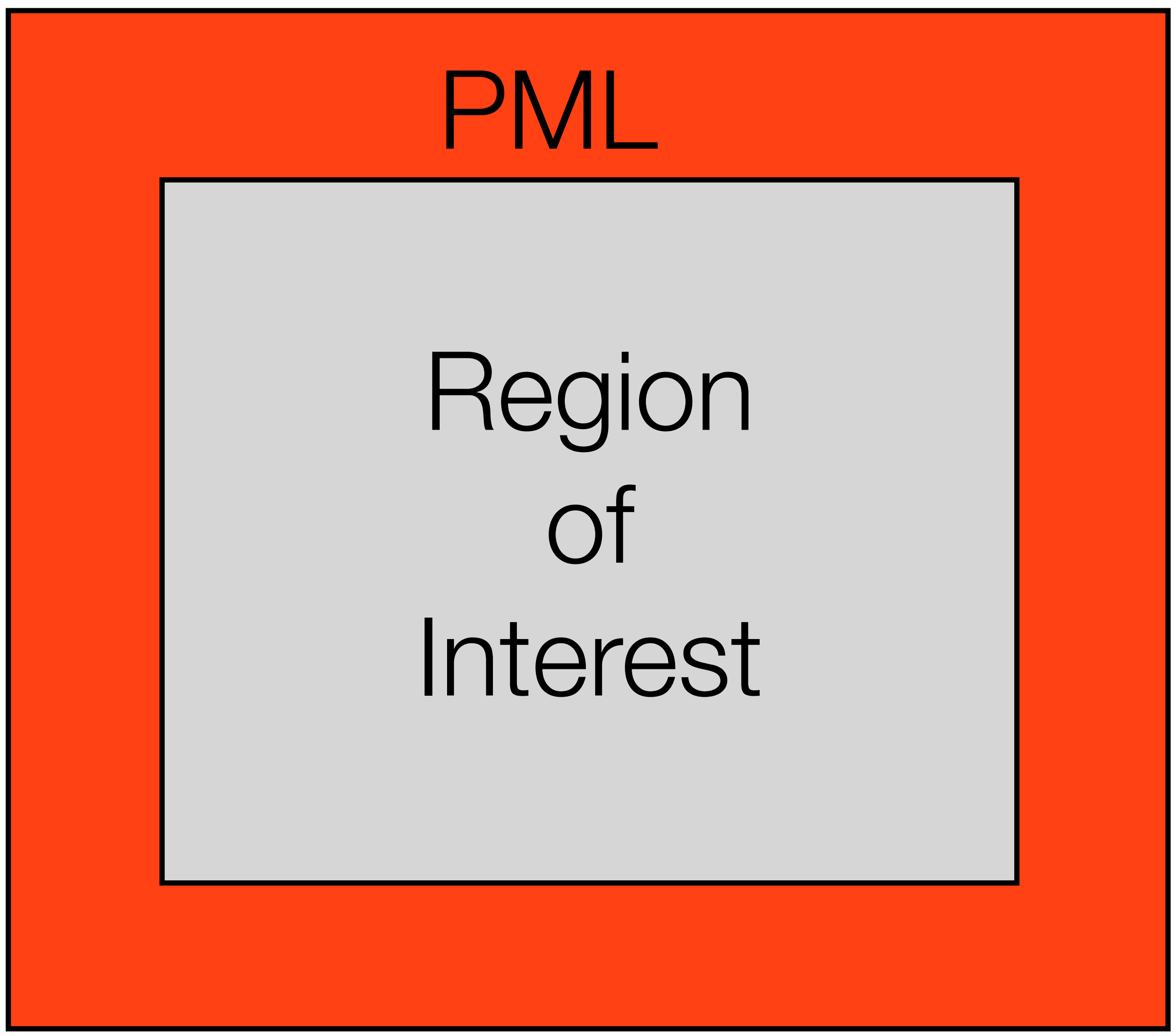
Quality is a MAC issue, not an accuracy one.

Looks better on Linux.

Possible Solution: Exact NRBC

- Dirichlet-to-Neumann boundaries: impose exact non-reflecting boundary conditions, constructed from Green's function to free wave.
- Nonlocal in time, nonlocal on boundary
- Internal solver restricted (no Fourier spectral methods)
- Geometry restricted
- Majda-Engquist, Bayliss-Turkell, Hagstrom, Greengard, Grote, ...

Possible Solution: Perfectly Matched Layers



- Extend with absorbing layer
- Dissipation inside layer
- Must be *Perfectly Matched* to avoid reflection at the interface.
- Equivalent to complex scaling

Possible Solution: Perfectly Matched Layers

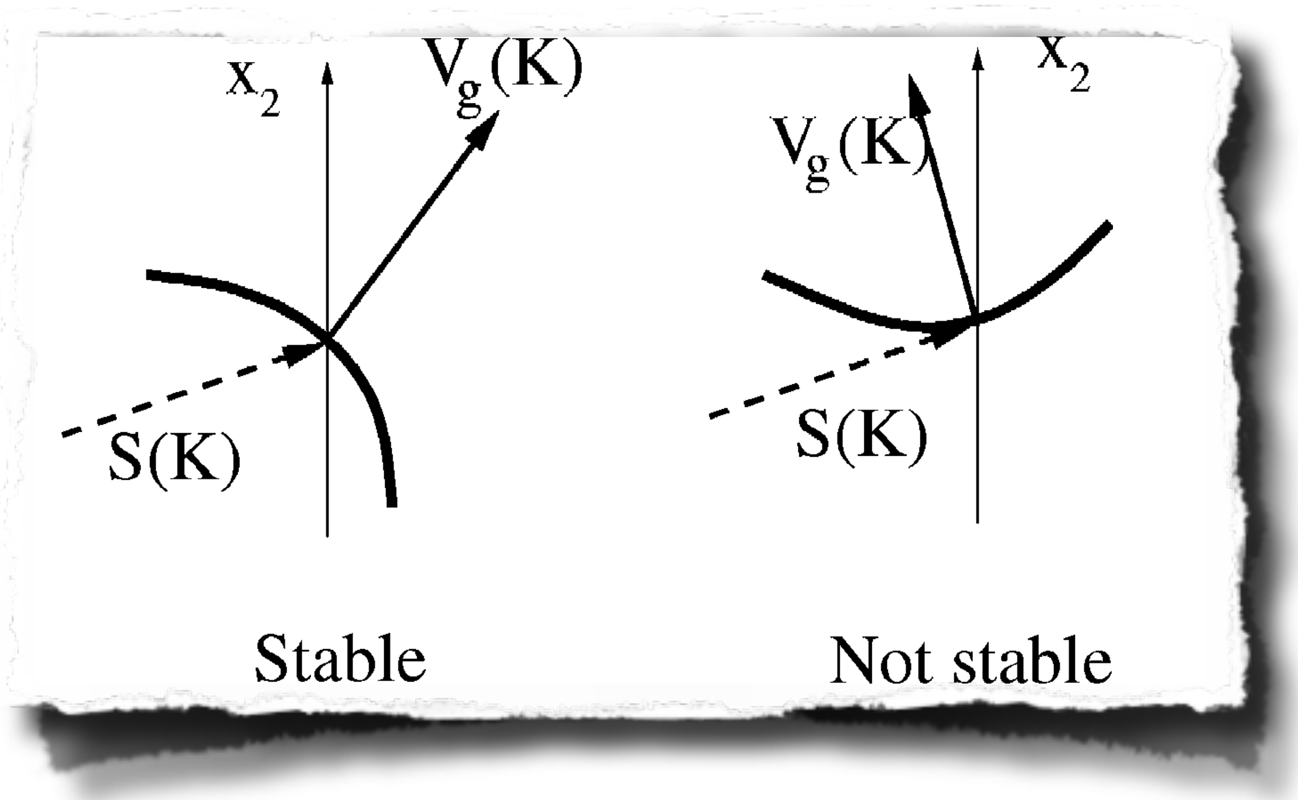
- Complex scaling for Wave equation:

$$\begin{aligned} H &\mapsto e^{zA} H e^{-zA} \\ A &= x \cdot i\nabla + i\nabla \cdot x \end{aligned}$$

- PML (Conjugate Operator) for general linear waves:

$$\begin{aligned} H &\mapsto e^{zA} H e^{-zA} \\ A &= x \cdot v_g(i\nabla) + v_g(i\nabla) \cdot x \end{aligned}$$

Complex scaling is easy



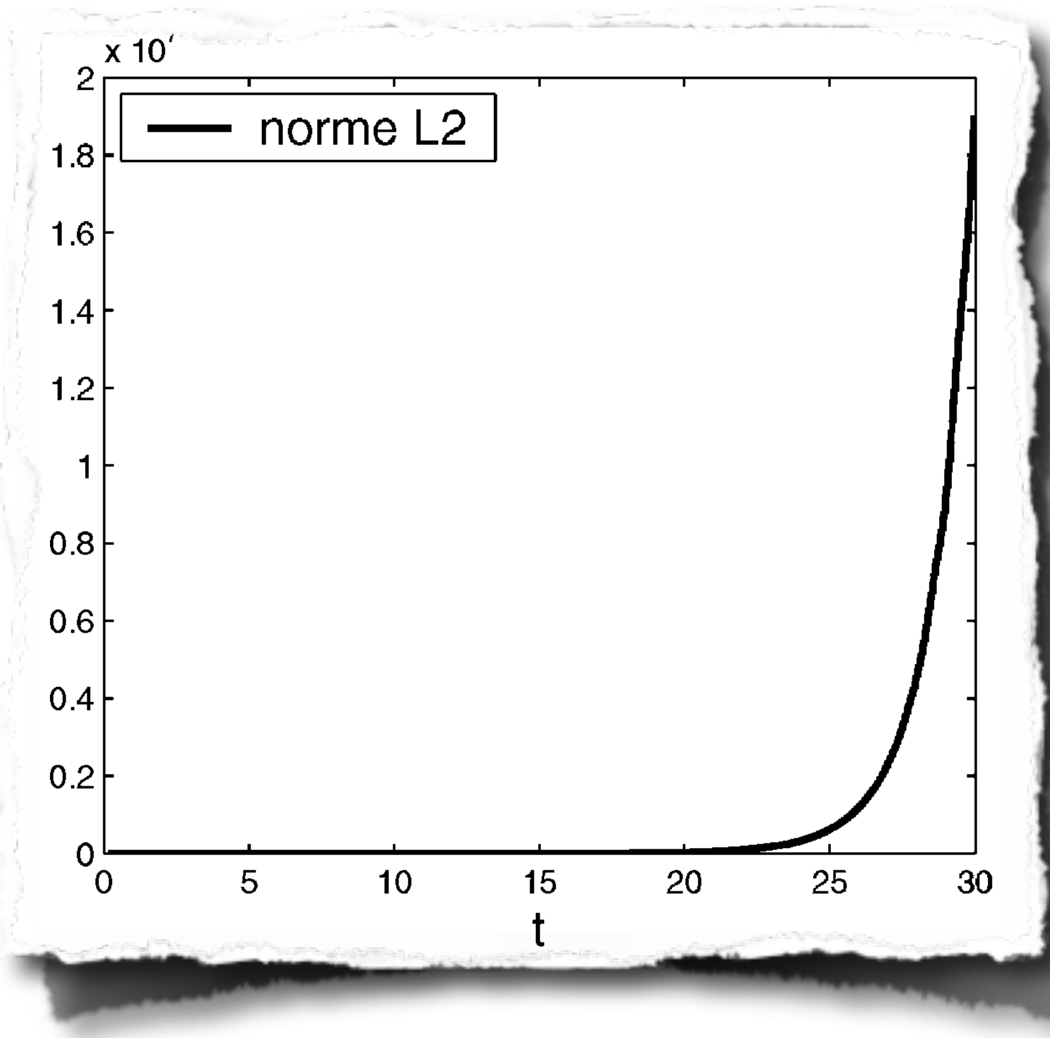
Picture from Becache, Fauqueux, Joly, JCP 188 (2003) 399–433.

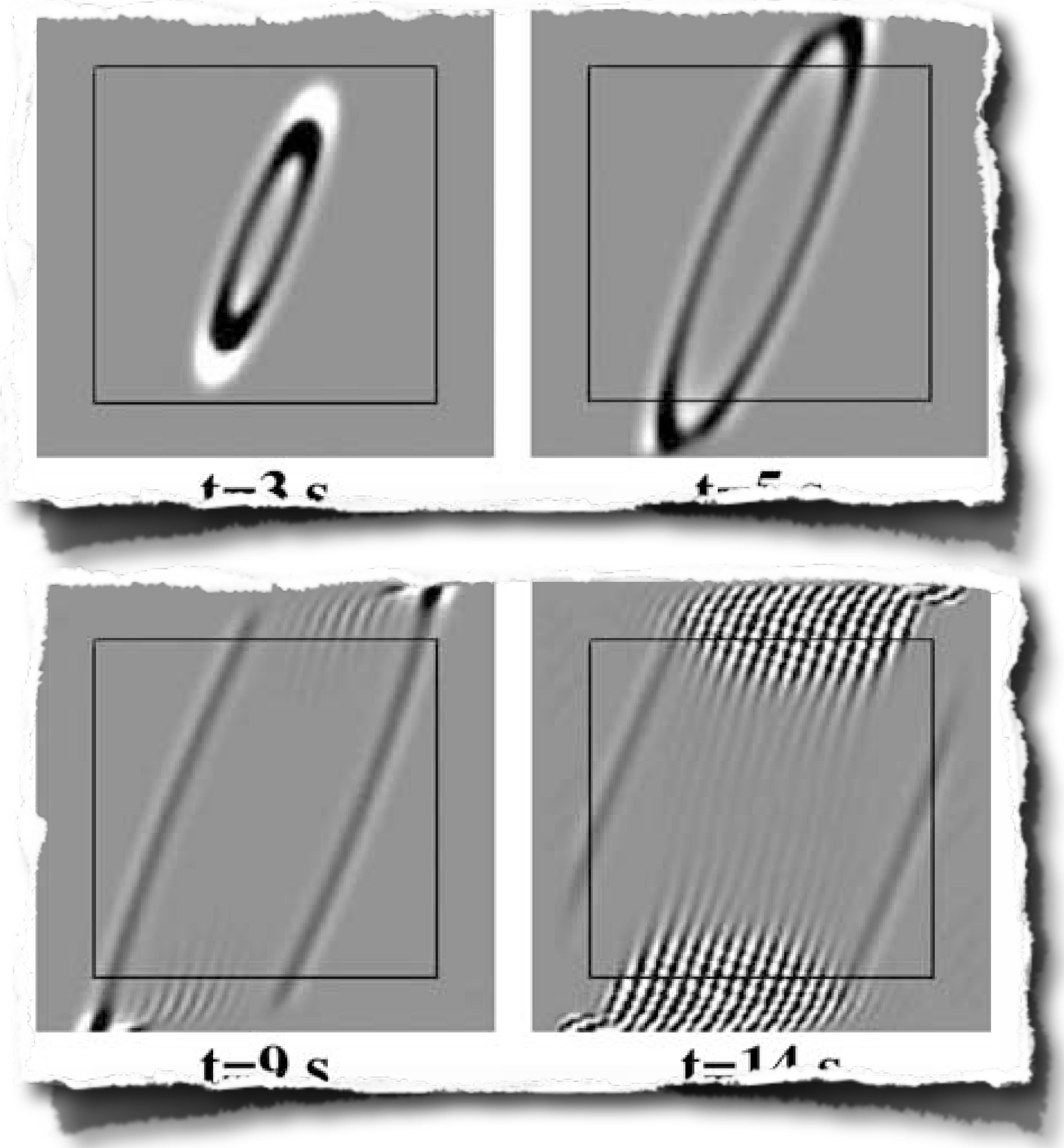
$$\begin{aligned} A &= x \cdot i\nabla + i\nabla \cdot x \\ e^{zA} &= \text{Dilation(z)} \end{aligned}$$

- Change coordinates
- Make layer perfectly matched
- Stable if $k_1 v_{g,1}(k) \geq 0$

PML Instability

- PML unstable for some anisotropic waves (Becache, Fauqueux, Joly, 2003).





Pictures from Becache, Fauqueux, Joly, JCP 188 (2003) 399–433.

Picture and graph are for different equations

Conjugate operators are hard

$$
\begin{aligned}
A &= x \cdot v_g(i\nabla) + v_g(i\nabla) \cdot x \\
e^{zA} &= ?
\end{aligned}
$$

If stability condition not satisfied, complex scaling shifts spectrum the wrong way.

Phase Space Filters

- Identify outgoing waves
- Filter them off
- Nothing hits the boundary

Outgoing waves

Outgoing Waves, Schrodinger Equation

- 1D Schrodinger Equation

$$\psi_0(x) = \frac{e^{ivx}}{\sqrt{\sigma}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

$$\psi(x,t) = \frac{e^{ivx}}{\sqrt{\sigma + it/\sigma}} \exp\left(\frac{-(x-vt)^2}{2\sigma^2(1+it/\sigma)}\right)$$

- Center of mass at $x = vt$, $\text{width} = \sigma + t/\sigma$

Outgoing Waves, Schrodinger Equation

- Outgoing wave

$$\psi_0(x) \quad = \quad e^{+ivx} e^{-(x-L)^2/\sigma^2}$$

$$\text{Trajectory} = L + vt$$

- Incoming wave

$$\psi_0(x) \quad = \quad e^{-ivx} e^{-(x-L)^2/\sigma^2}$$

$$\text{Trajectory} = L - vt$$

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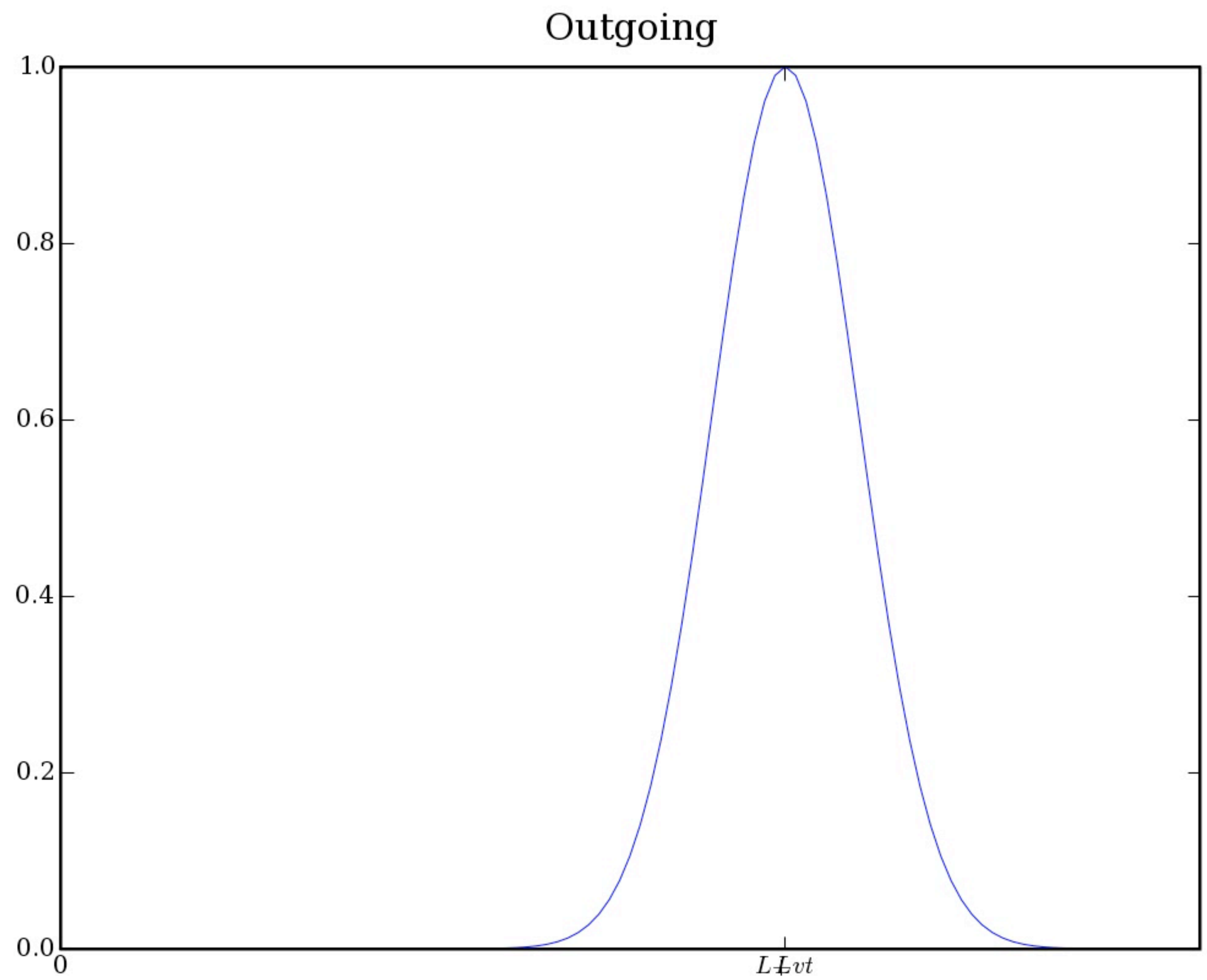

Outgoing Waves, Schrodinger Equation

- Outgoing wave

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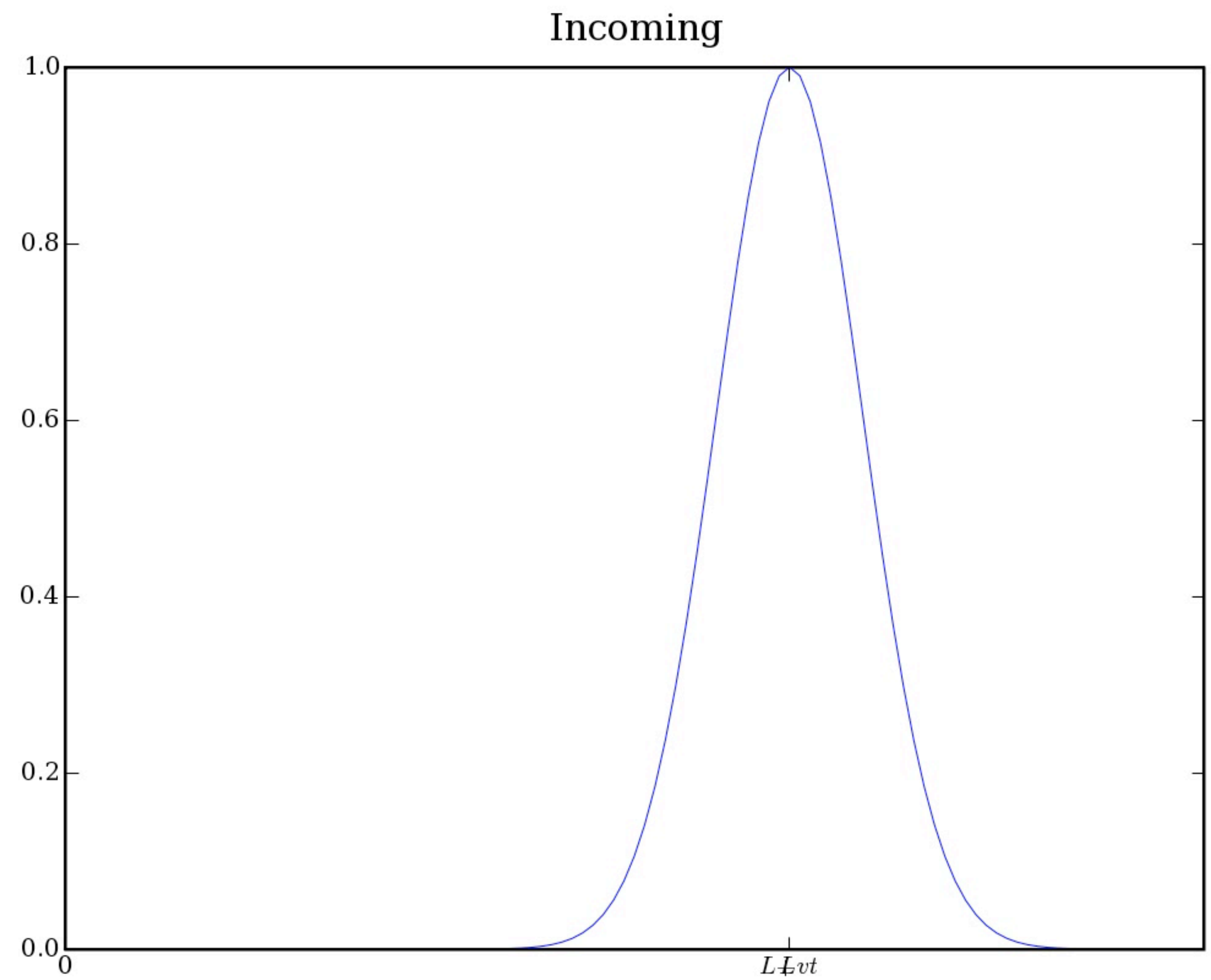


Outgoing Waves, Schrodinger Equation

- Mixed wave:

$$\psi_0(x) \quad = \quad e^{-ivx} e^{-(x-L)^2/\sigma^2} + e^{+ivx} e^{-(x-L)^2/\sigma^2}$$

Outgoing Waves, Schrodinger Equation

- Mixed wave:

$$\psi_0(x) \quad = \quad e^{-ivx} e^{-(x-L)^2/\sigma^2} + e^{+ivx} e^{-(x-L)^2/\sigma^2}$$

Incoming wave

Outgoing wave

Outgoing Waves, Schrodinger Equation

- Mixed wave:

$$\psi_0(x) \quad = \quad e^{-ivx}e^{-(x-L)^2/\sigma^2} \qquad +0$$

Incoming wave

Outgoing wave

Outgoing Waves, Schrodinger Equation

- Mixed wave:

$$\psi_0(x) \quad = \quad e^{-ivx} e^{-(x-L)^2/\sigma^2}$$

Incoming wave

Problem solved!

It really is that easy

- Windowed Fourier Transform:

$$\psi(x) = \sum_{a \in Z} \sum_{b \in Z} \psi_{a,b} e^{ibk_0 x} g(x - ax_0)$$

$$g(x) = e^{-x^2/\sigma^2}$$

- Outgoing waves:

$$\begin{aligned} ax_0 &> L \\ bk_0 &> \sigma^{-1} \end{aligned}$$

Theorem 2.5: If

1) $m(g;q_0) = \operatorname*{ess\,inf}_{x \in [0,q_0]} \sum_n |g(x - nq_0)|^2 > 0$ (2.3.11)

2) $M(g;q_0) = \operatorname*{ess\,sup}_{x \in [0,q_0]} \sum_n |g(x - nq_0)|^2 < \infty$ (2.3.12)

and

3) $\sup_{s \in \mathbb{R}} \left[(1+s^2)^{(1+\epsilon)/2} \beta(s)\right] = C_\epsilon < \infty$ for some $\epsilon >$ (

where

$$\beta(s) = \sup_{x \in [0,q_0]} \sum_{n \in \mathbb{Z}} |g(x - nq_0)| \, |g(x + s - nq_0)|$$

hen there exists a $P_0^c > 0$ such that

$\forall p_0 \in (0, P_0^c)$: the g_{mn} associated with g, p_0, q_0 are a frame

$\forall \delta > 0 : \exists p_0$ in $[P_0^c, P_0^c + \delta]$ such that the g_{mn} associated to g, p_0, q_0 are not a frame.

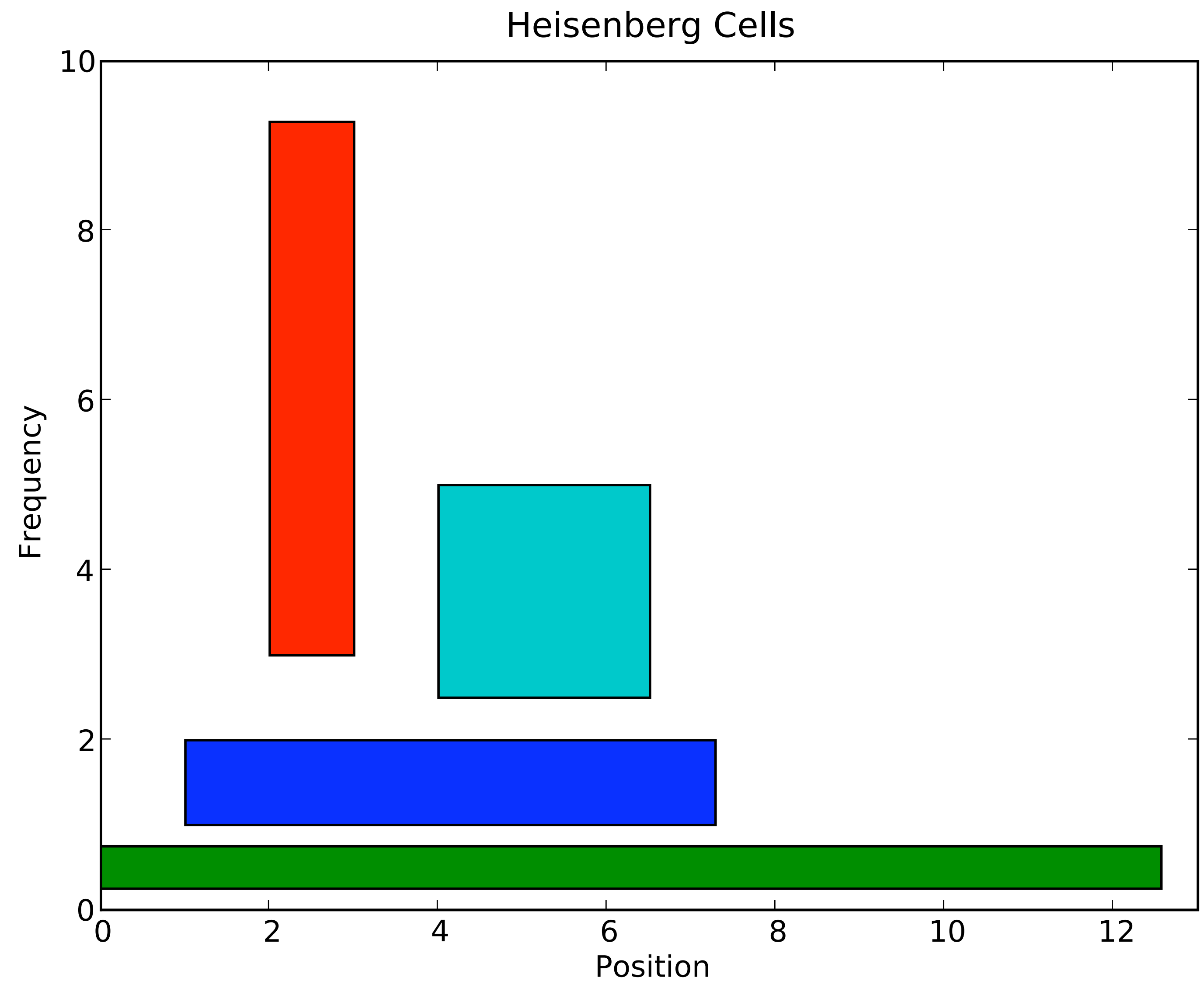
The Wavelet Transform, Time Frequency Localization and Signal Analysis, Ingrid Daubechies, IEEE Trans. Info. Theory, Vol 36 **5** 1990

Quantum Phase Space

- Quantum phase space is set of points $(x, k) \in \mathbb{R}^N \times \mathbb{R}^N$, x a position and k a frequency.

- Heisenberg Uncertainty principle: localizing on region of volume $O(2\pi \ln(\epsilon))$ causes error ϵ .

- A function is localized near a point (x_0, k_0) if it is localized in position near x_0 and it's Fourier transform is localized near k_0 .



Outgoing Waves, Schrodinger Equation

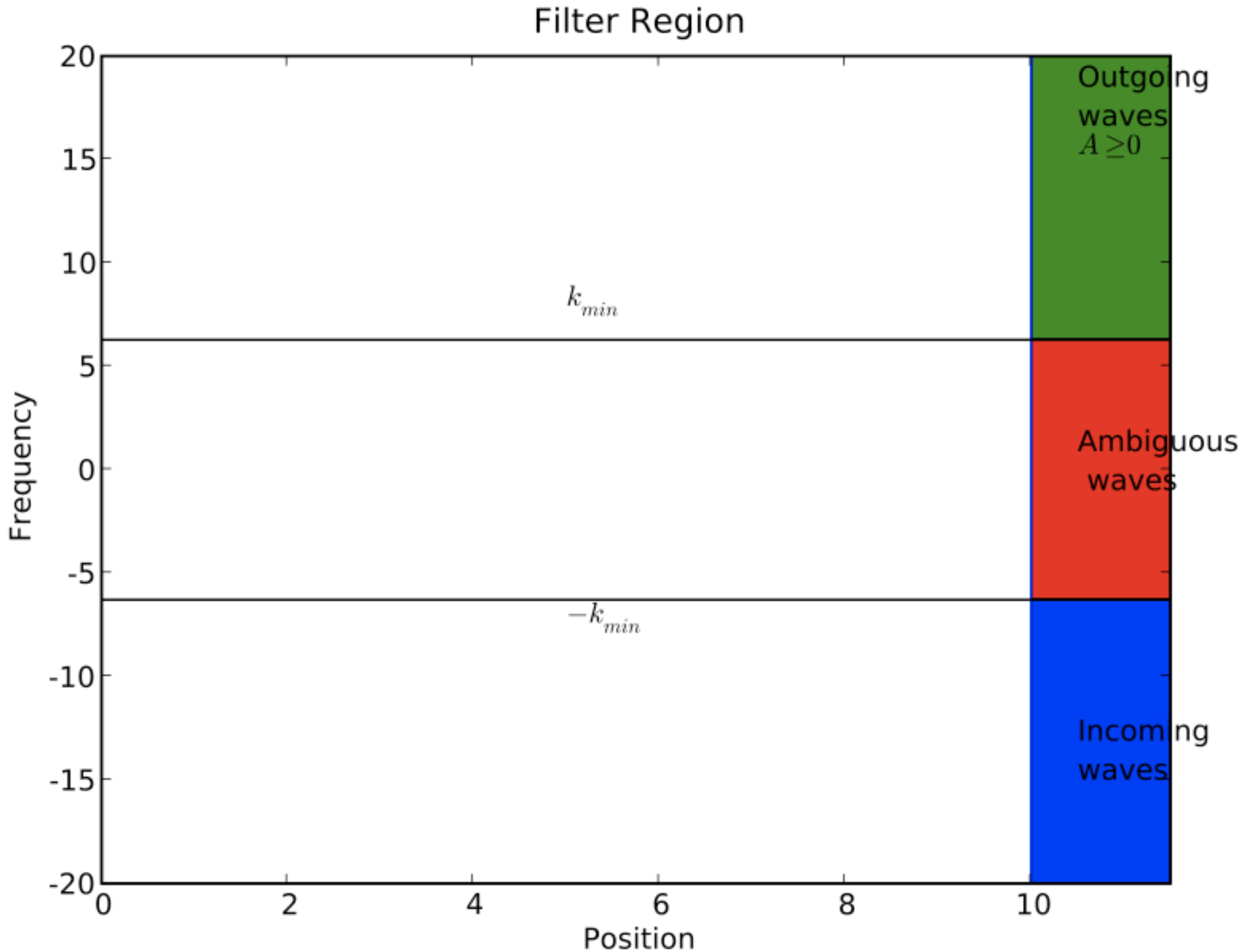
- Ambiguous waves

$$\psi_0(x) = e^{i0x} e^{-(x-L)^2/\sigma^2}$$

- Spreads in both directions

Issue can be resolved.

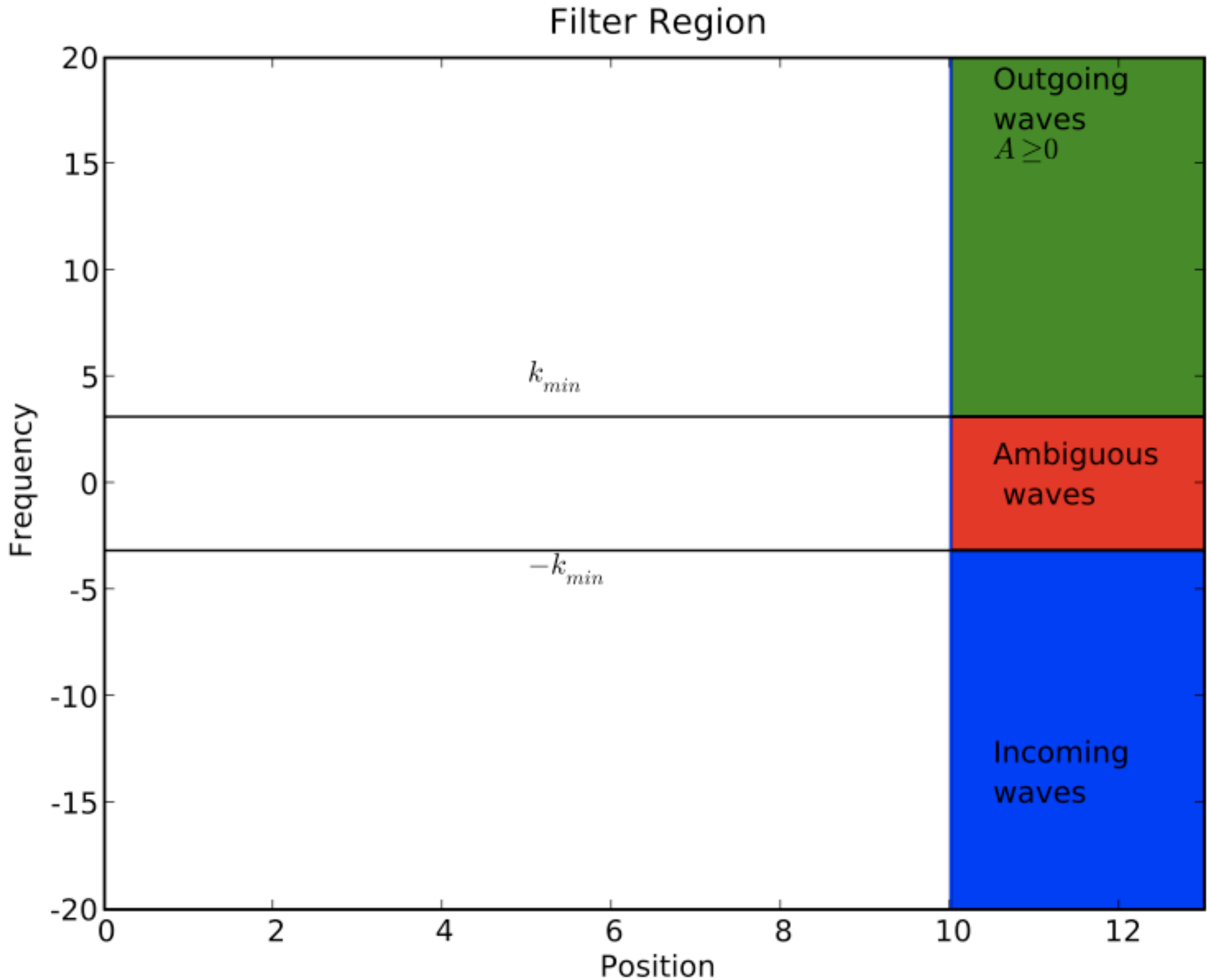
Phase space filters



Phase Space Filters

Outgoing Waves:

$$ax_0 > L$$
$$bk_0 > \sigma^{-1}$$



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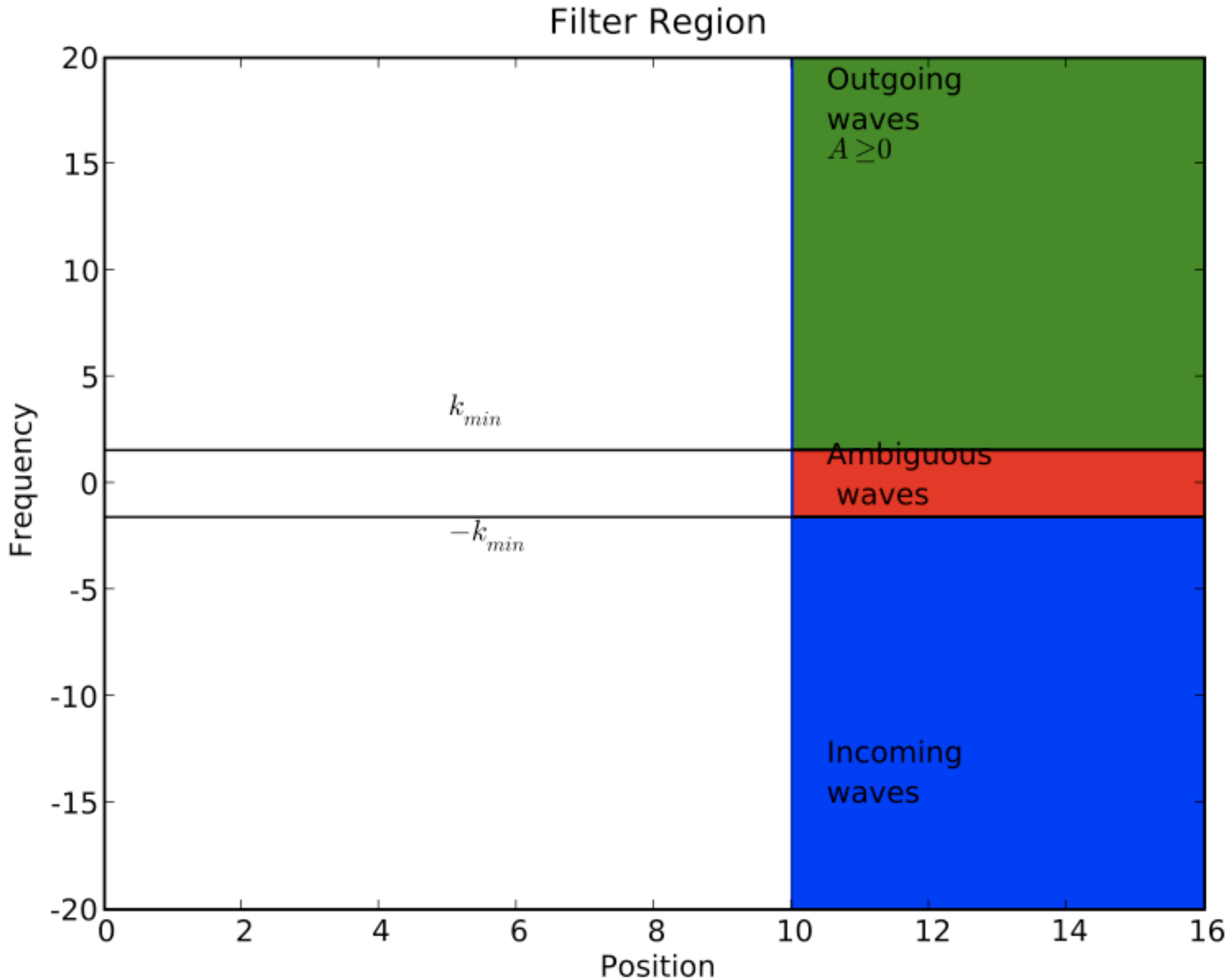


Phase Space Filters

Outgoing Waves:

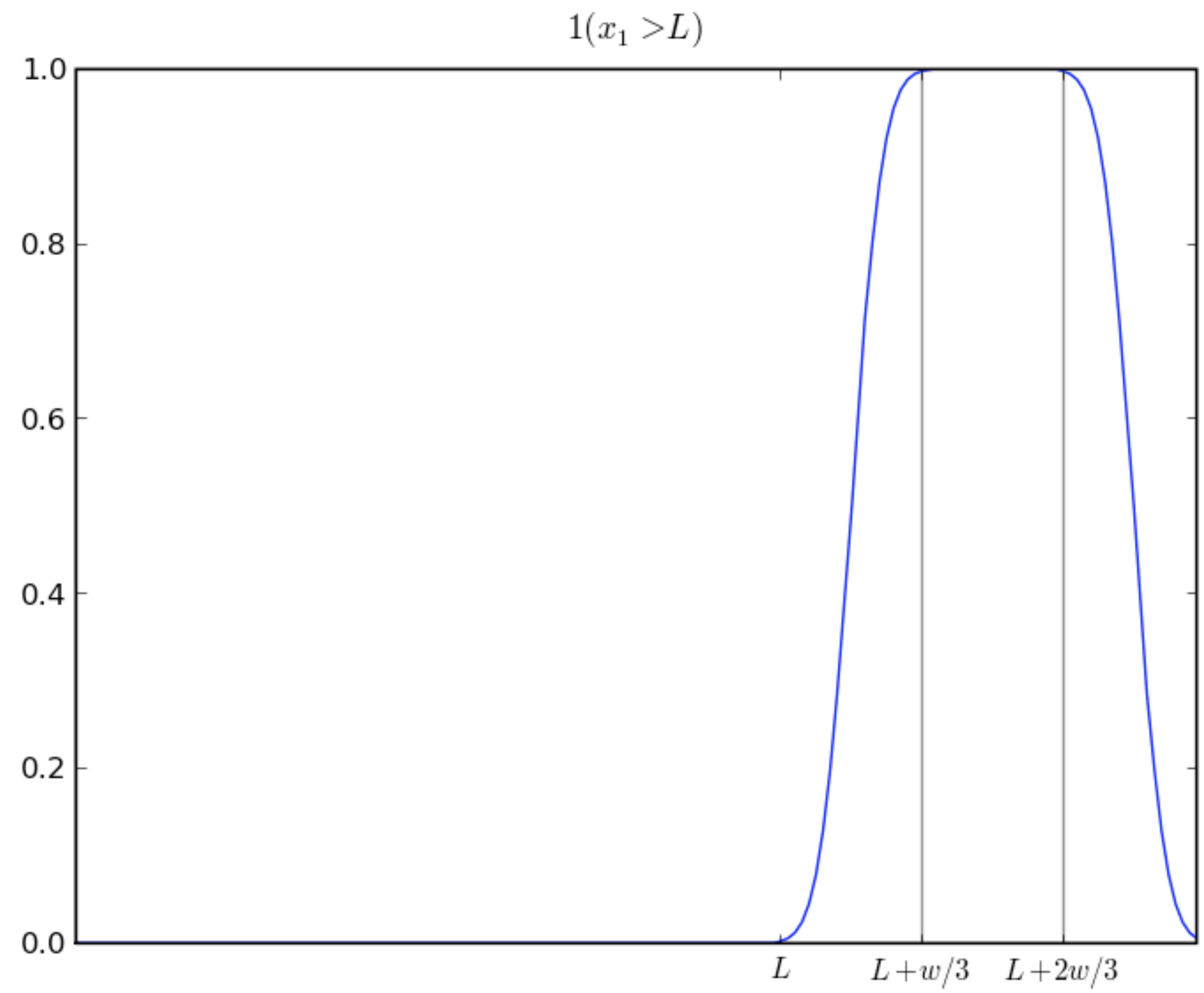
$$ax_0 > L$$
$$bk_0 > \sigma^{-1}$$

A simpler version

- We want rightward moving waves *near the boundary*.
- Extend computational domain

$$[-L-w, L+w]^N$$

- Localize in boundary layer



$$1(x > L)1(k > k_{min})1(x > L) = O^{+}$$

Step function built out of erfc(x) functions, as smooth as possible without overflowing.

How does it work?

- Take wave comprised of incoming and outgoing waves, plus interior waves.

$$\begin{aligned} \psi_0(x) &\approx e^{ivx} g(x-L-1) + e^{-ivx} g(x-L-1) \\ &+ \text{interior waves} \end{aligned}$$

How does it work?

- Take wave comprised of incoming and outgoing waves, plus interior waves.

$$\psi_0(x) \approx e^{ivx} g(x-L-1) + e^{-ivx} g(x-L-1) + \text{interior waves}$$

Our target

How does it work?

- Take wave comprised of incoming and outgoing waves, plus interior waves.

$$\begin{aligned}\psi_0(x) &\approx e^{ivx} g(x-L-1) + e^{-ivx} g(x-L-1) \\ &+ \text{interior waves}\end{aligned}$$

- We don’t care about interior waves

$$\begin{aligned}1(x>L)\psi_0(x) &\approx e^{ivx} g(x-L-1) + e^{-ivx} g(x-L-1) \\ &+ 0\end{aligned}$$

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- Or incoming waves

$$1(k>k_{min})1(x>L)\psi_0(x) \approx e^{ivx} g(x-L-1) + 0$$

How does it work?

- Or incoming waves

$$1(k > k_{min})1(x > L)\psi_0(x) \quad \approx \quad e^{ivx}g(x - L - 1) + 0$$

- Symmetry is always good (for stability, etc):

$$1(x > L)1(k > k_{min})1(x > L)\psi_0(x) \quad \approx \quad e^{ivx}g(x - L - 1)$$

How does it work?

- Or incoming waves

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- Symmetry is always good (for stability, etc):

$$1(x > L)1(k > k_{min})1(x > L)\psi_0(x) \quad \approx \quad e^{ivx}g(x - L - 1)$$

- Operator O^+ localizes outgoing waves, and lets us remove them:

$$\psi_0(x) - O^+\psi_0(x) = 0 + e^{-ivx}g(x - L) + \text{Interior Waves}$$

Propagation Algorithm

let $T_s := O(w/3v_{max}\ln(\epsilon))$

let $u(x) := u_0(x)$ on domain $[-L-w, L+w]^N$

for n = 1 to T_{max}/T_s:

$$u(x) \leftarrow e^{i\Delta T_s}u(x)$$

$$u(x) \leftarrow \left[\prod_{\text{all sides}}(1-O^+)\right]u(x)$$

output $u(x) = u(x, nT_s)$

Propagation Algorithm

Not enough time to travel distance w

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Propagate any way you like.

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(Nothing reached the boundary yet.)

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output $u(x) = u(x, nT_s)$

Filter outgoing waves about to reach the boundary.

Propagation Algorithm

Not enough time to travel distance w

let $T_s := O(w/3v_{max}\ln(\epsilon))$

let $u(x) := u_0(x)$ on domain $[-L-w, L+w]^N$

Propagate any way you like.

(Nothing reached the boundary yet.)

Next propagation step is accurate: waves which would have reached boundary were filtered.

for n = 1 to T_{max}/T_s:

$$u(x) \leftarrow e^{i\Delta T_s} u(x)$$

$$u(x) \leftarrow \left[\prod_{\text{all sides}} (1 - O^+)\right] u(x)$$

output $u(x) = u(x, nT_s)$

Filter outgoing waves about to reach the boundary.

Phase Space Filtering, Schrodinger Equation

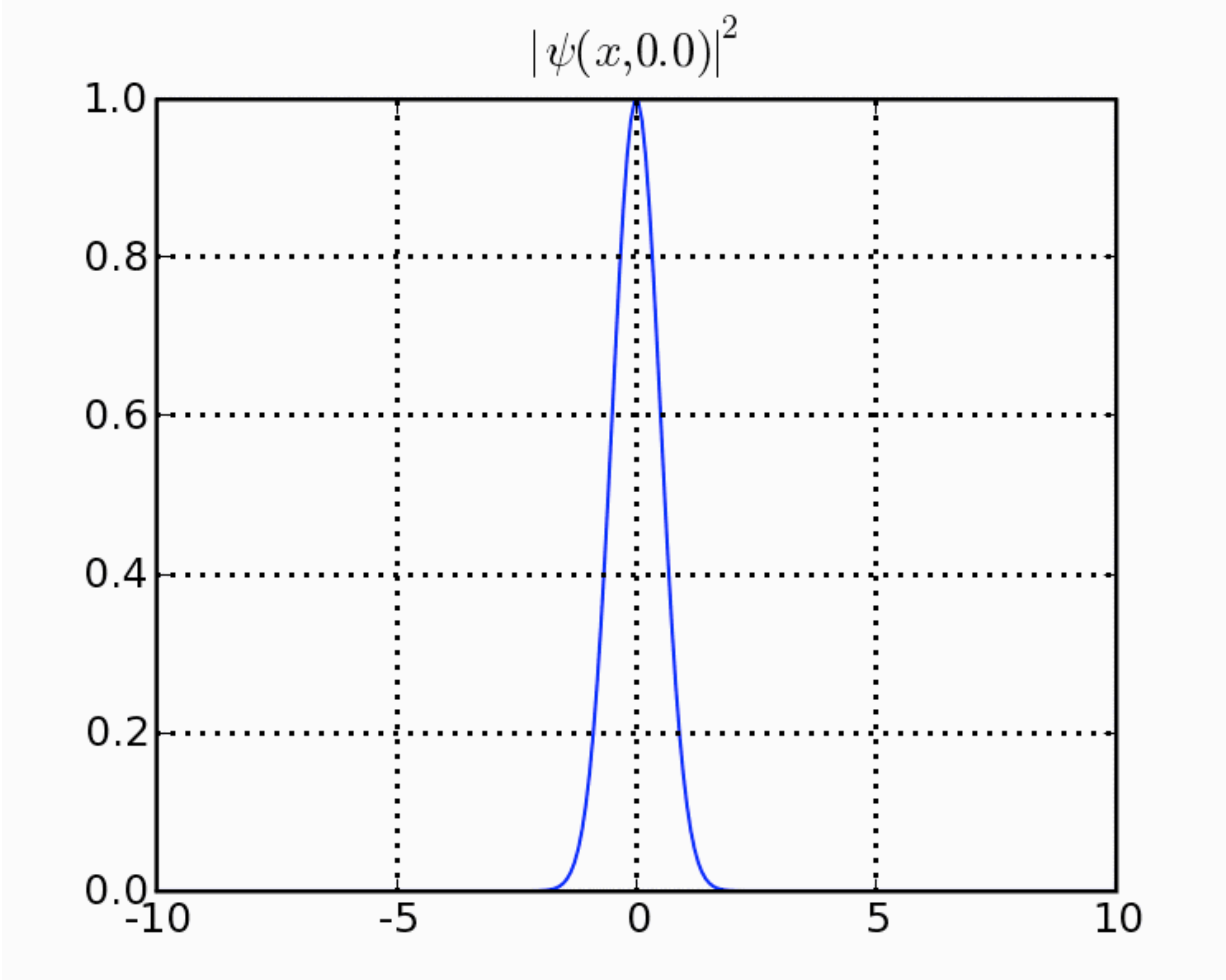
$$O^{+} = 1(x_1 > L)1(k > k_{min})1(x_1 > L)$$

- $1(x_1 > L)$ is "blurring" operator in frequency domain

$$[\widehat{1(x_1 > L)}f](k) \approx (...)e^{-k^2/w^2} \star \hat{f}(k)$$

- Characteristic distance of "blurring" (in k domain)

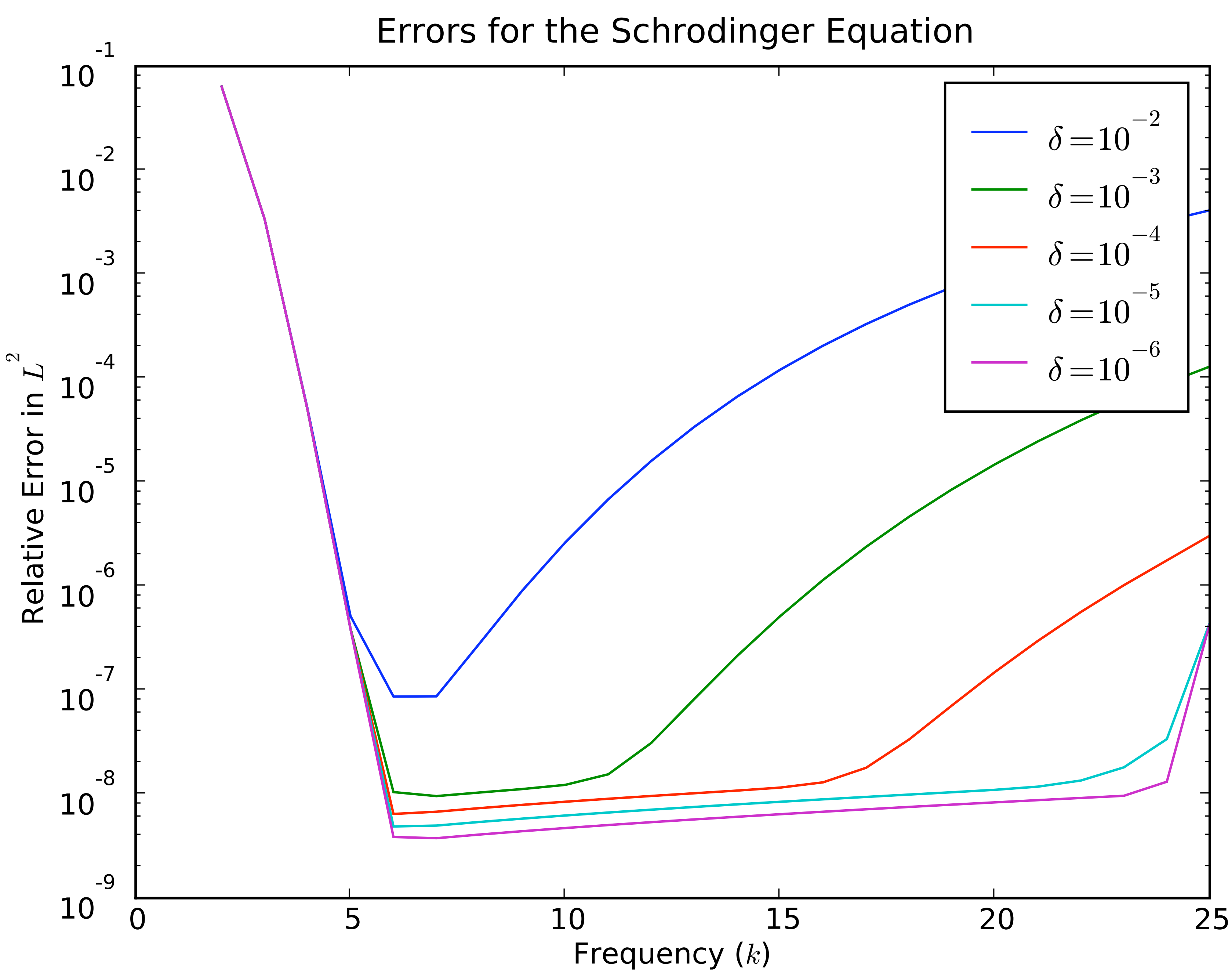
$$k_{min} = O(\ln(\epsilon^{-1})/w)$$



Schrodinger Equation

Results

Schrodinger equation: Error vs Frequency



- Measured error as function of frequency of initial data.
- Errors are large for low frequencies, small for high.
- By increasing width of buffer, one reduce errors for low frequencies.

Phase Space Filters for Vector Systems

Vector Systems

- Linear wave equation:

$$\begin{aligned} \vec{u}_t(x,t) &= H\vec{u}(,t) \\ H(i\nabla) &= -H^\dagger(i\nabla) \end{aligned}$$

$$H = \left[\begin{array}{lll} H_{11}(k) & \ldots & H_{1N}(k) \\ \ldots & \ldots & \ldots \\ -H_{1N}(k) & \ldots & H_{NN}(k) \end{array} \right]$$

Wavepackets

- Not a 1-way wavepacket:

$$u_0(x) = \begin{bmatrix} e^{ikx}g(x) \\ \dots \\ 0 \end{bmatrix}$$

- Will split into N different wavepackets.

Wavepackets

- Diagonalize hamiltonian to find dispersion relation

$$H = D^\dagger \begin{bmatrix} i\omega_1(k) & \dots & 0 \\ \dots & i\omega_j(k) & \dots \\ 0 & \dots & i\omega_M(k) \end{bmatrix} D$$

- For each frequency, H is skew adjoint matrix. Can always do this.

- Plane Waves:

$$h(x,t) = \begin{bmatrix} d_{1,1}(k) \\ \dots \\ d_{1,N}(k) \end{bmatrix} e^{i(kx - \omega_1(k)t)}$$

Wavepackets

- Localize a plane wave:

$$u_0(x) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{ik_0 x} g(x)$$

Wavepackets

- Localize a plane wave:

$$u_0(x) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{ik_0 x} g(x)$$

- Wavepacket propagation:

$$u(x,t) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{i(k_0 x - \omega_1(k_0)t)} [e^{Dt} g](x - \nabla_k \omega_1(k_0)t)$$

- (Fourier transform and do stationary phase)

Wavepackets

- Localize a plane wave:

$$u_0(x) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{ik_0 x} g(x)$$

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Translation

Wavepackets

- Localize a plane wave:

$$u_0(x) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{ik_0 x} g(x)$$

Dispersion

- Wavepacket propagation:

$$u(x,t) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{i(k_0 x - \omega_1(k_0)t)} [e^{Dt} g](x - \nabla_k \omega_1(k_0) t)$$

- (Fourier transform and do stationary phase)

Translation

Wavepackets

- Envelope obeys Schrodinger like equation:

$$
\begin{aligned}
\widehat{[e^{Dt}g]}(k) &= \exp((\omega_q(k) - \omega_1(k_0) - [\nabla_k(\omega_q)](k_0)(k-k_0))t)\hat{g}(k) \\
&\approx e^{(k-k_0)[H\omega_1(k_0)](k-k_0)t}\hat{g}(k)
\end{aligned}
$$

- $H\omega_1(k_0)$ is the Hessian of the dispersion relation.

Hessian is Quadratic Differential operator, like Laplacian.

Wavepackets

- Schrodinger:

$$\begin{aligned} \psi_0(x) &= e^{ikx} e^{-x^2/\sigma^2} \\ \text{position} &= kt \end{aligned}$$

- Vector system:

$$u_0(x) = \begin{bmatrix} d_{11}(k_0) \\ \dots \\ d_{1N}(k_0) \end{bmatrix} e^{ik_0 x} g(x)$$

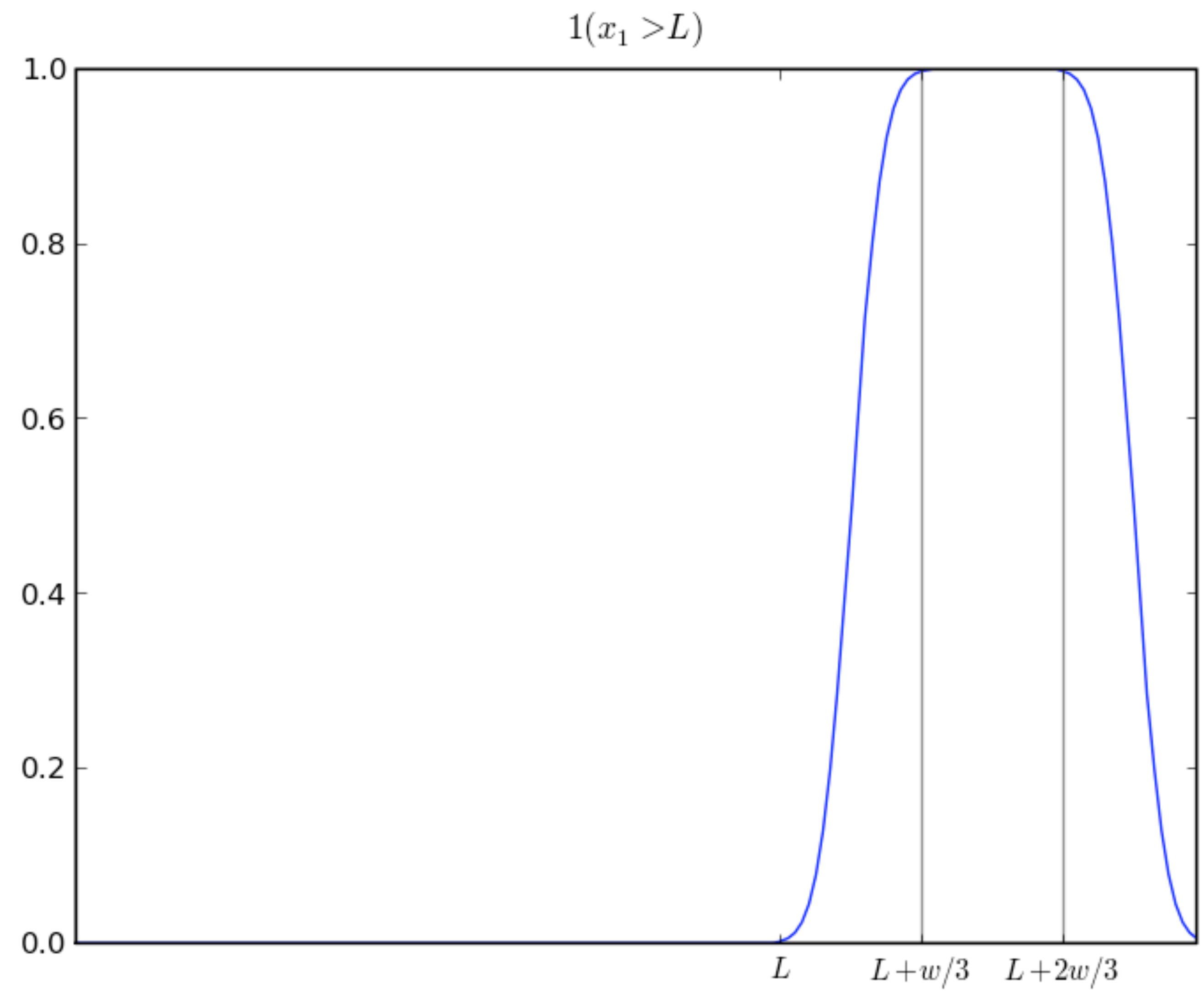
$$\text{position} = [\nabla_k \omega_1(k_0)]t = v_g(k_0)t$$

Phase space filtering for vector systems

- We want rightward moving waves *near the boundary*.
- Extend computational domain

$$[-L-w, L+w]^N$$

- Localize in boundary layer



Step function built out of erfc(x) functions, as smooth as possible without overflowing.

Phase space filtering for vector systems

- Project onto rightward-moving group velocities

$$P(k) = \begin{bmatrix} 1(\nabla_k \omega_1(k) \cdot e_1 > 0) & \dots & 0 \\ ... & \dots & \\ 0 & \dots & 1(\nabla_k \omega_M(k) \cdot e_1 > 0 \end{bmatrix}$$

- Un-diagonalize to project onto rightward moving waves

$$D^\dagger P(k) D$$

- Localize:

$$O^+ = 1(x_1 > L) D^\dagger P(k) D 1(x_1 > L)$$

Propagation Algorithm

let $T_s = O(w/3v_{max}\ln(\epsilon))$,

let $u(x) := u_0(x)$ on domain $[-L-w, L+w]^N$

for n = 1 to T_{max}/T_s:

$$u(x) \leftarrow e^{iHT_s}u(x)$$

$$u(x) \leftarrow \left[\prod_{\text{all sides}}(1-O^+)\right]u(x)$$

output $u(x) = u(x, nT_s)$

Numerical Results, Anisotropic Waves

Maxwell's Equations in Birefringent Medium

- In a birefringent medium, Maxwell's equations take the form

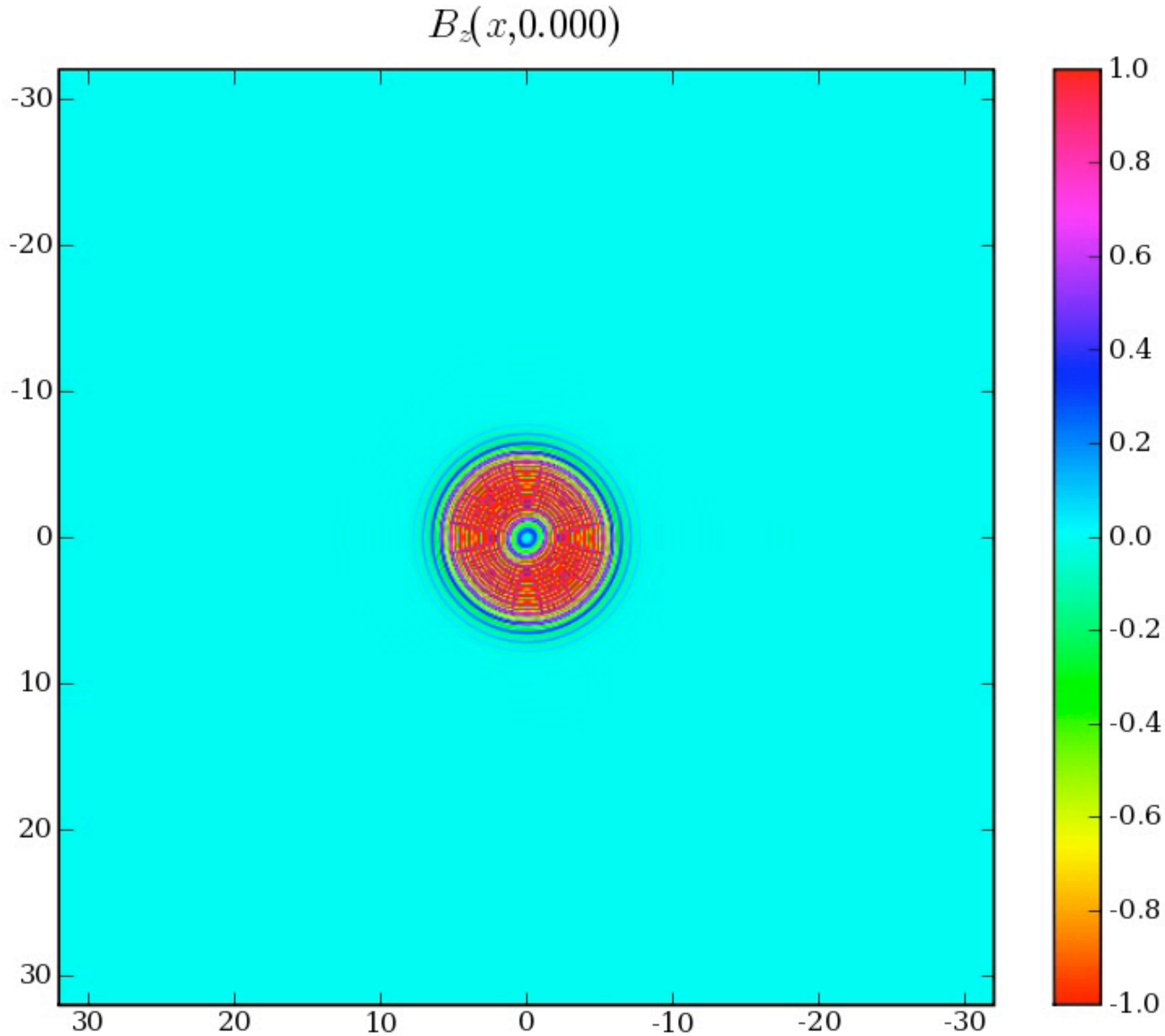
$$H = \begin{bmatrix} 0 & -\mu^{-1/2}\nabla \times \epsilon^{-1/2} \\ \epsilon^{-1/2}\nabla \times \mu^{-1/2} & 0 \end{bmatrix}$$

- The wavefield is defined as $u(x,t) = (\sqrt{\mu}\vec{H}, \sqrt{\epsilon}\vec{E})^T$

- Assume μ is a scalar, and assume $\epsilon = \begin{bmatrix} 1 & b & 0 \\ b & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$

- Then with $f = (1/2)(\sqrt{1+b} + \sqrt{1-b}), g = (1/2)(-\sqrt{1+b} + \sqrt{1-b}),$

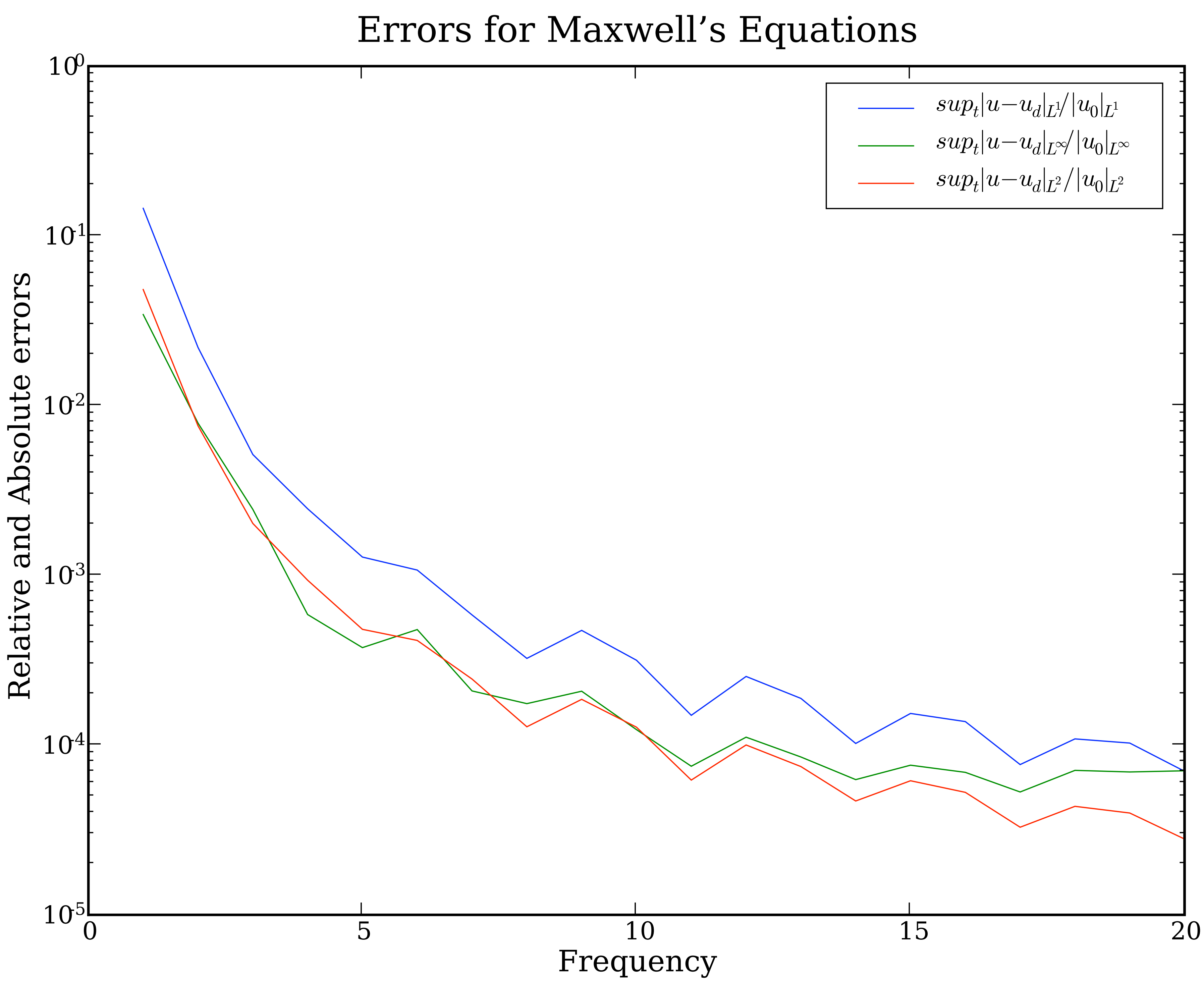
$$\begin{aligned} \omega_{j=1,2}(k) &= (-1)^{1+j} i c^{-1}|k| \\ \omega_{j=3,4}(k) &= (-1)^{1+j} i \sqrt{(f^2+g^2)(k_1^2+k_2^2) - 4fgk_1k_2} \\ \omega_{j=5,6}(k) &= 0. \end{aligned}$$



Maxwell's Equations, TM_z mode

Birefringent medium, b=0.25

Maxwell’s Equations: Error vs Frequency



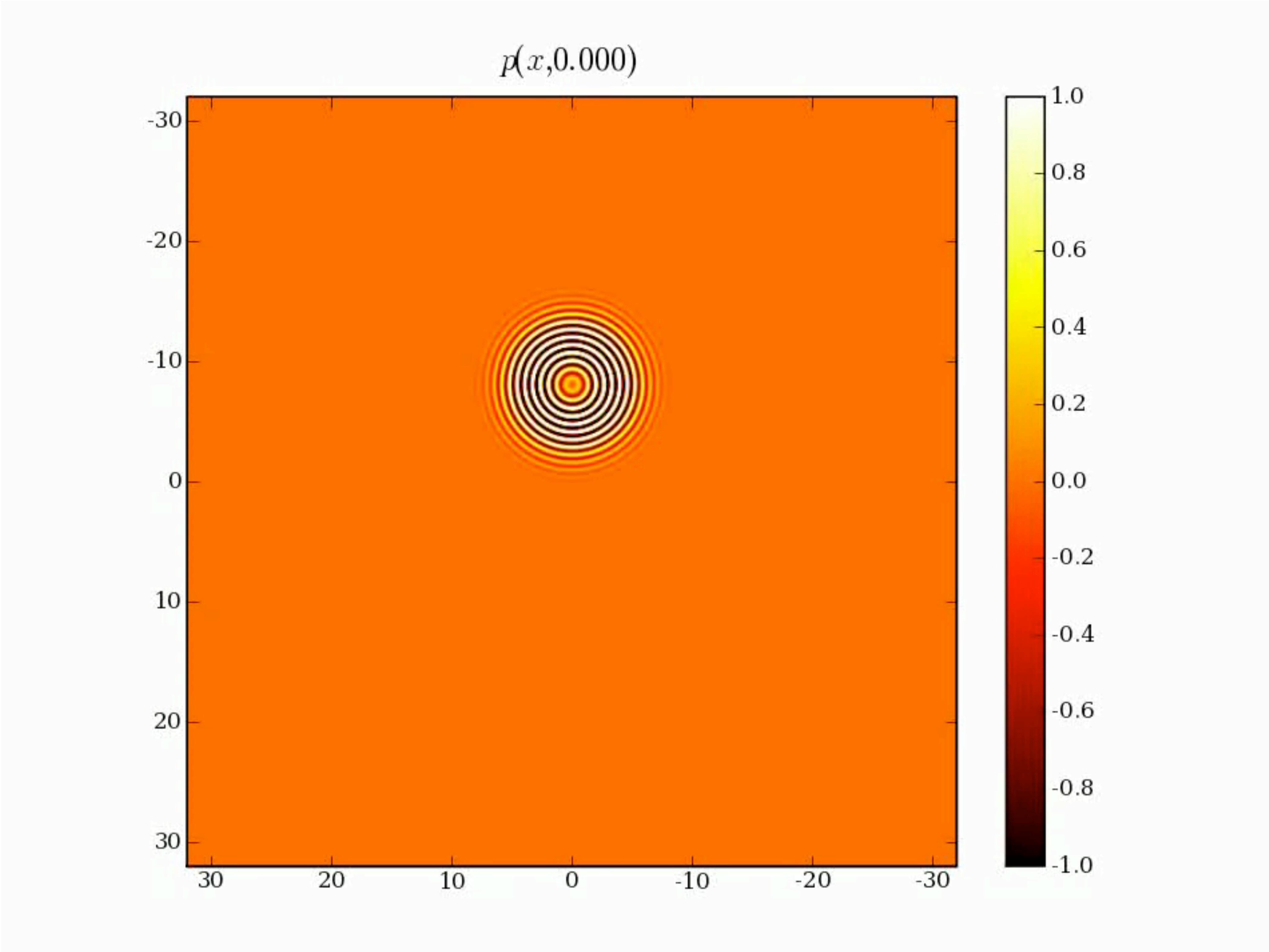
Linearized Euler Equations

- Euler equations, linearized about jet flow:

$$H = \begin{bmatrix} M\partial_{x_1} & -\partial_{x_1} & -\partial_{x_2} \\ -\partial_{x_1} & M\partial_{x_1} & 0 \\ -\partial_{x_2} & 0 & M\partial_{x_1} \end{bmatrix}$$

- The dispersion relations are:

$$\begin{aligned} \omega_1(k) &= Mk_1 + |k|, \\ \omega_2(k) &= Mk_1 - |k|, \\ \omega_3(k) &= Mk_1 \end{aligned}$$



Euler Equations

Results, M=0.5

Linearized Quasi-Geostrophic Equations

- Quasi-geostrophic equations, midlatitude:

$$H = V\partial_x - \tilde{\beta}(-\Delta + F)^{-1}\partial_x$$

$$V = \text{Mean wind}, F \sim \frac{(\text{earth's rotation})^2}{g}$$

$$\tilde{\beta} = FV + \beta, \beta = R\cos(\phi)$$

- ψ is a streamfunction: $\vec{v} = \nabla^{\perp}\psi$

- Geostrophic balance: Coriolis force = horizontal pressure gradient

- Anisotropic and non-local

\phi = latitude.

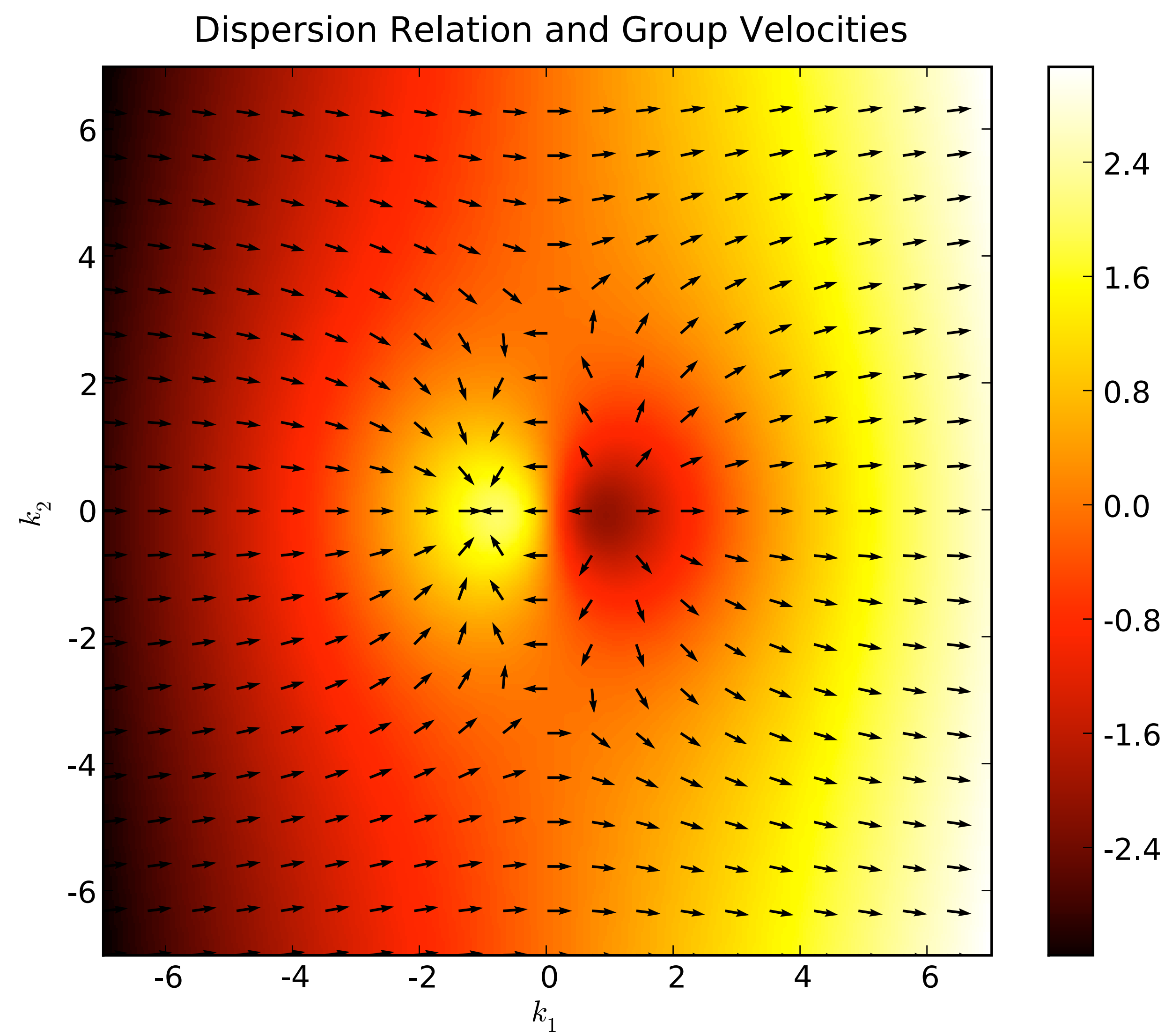
Mention that Tom Hagstrom prompted this example.

Dispersion Relations

- Complicated dispersion relation, not quite hyperbolic

- PML unstable in y direction for

$$k_0 < 0$$

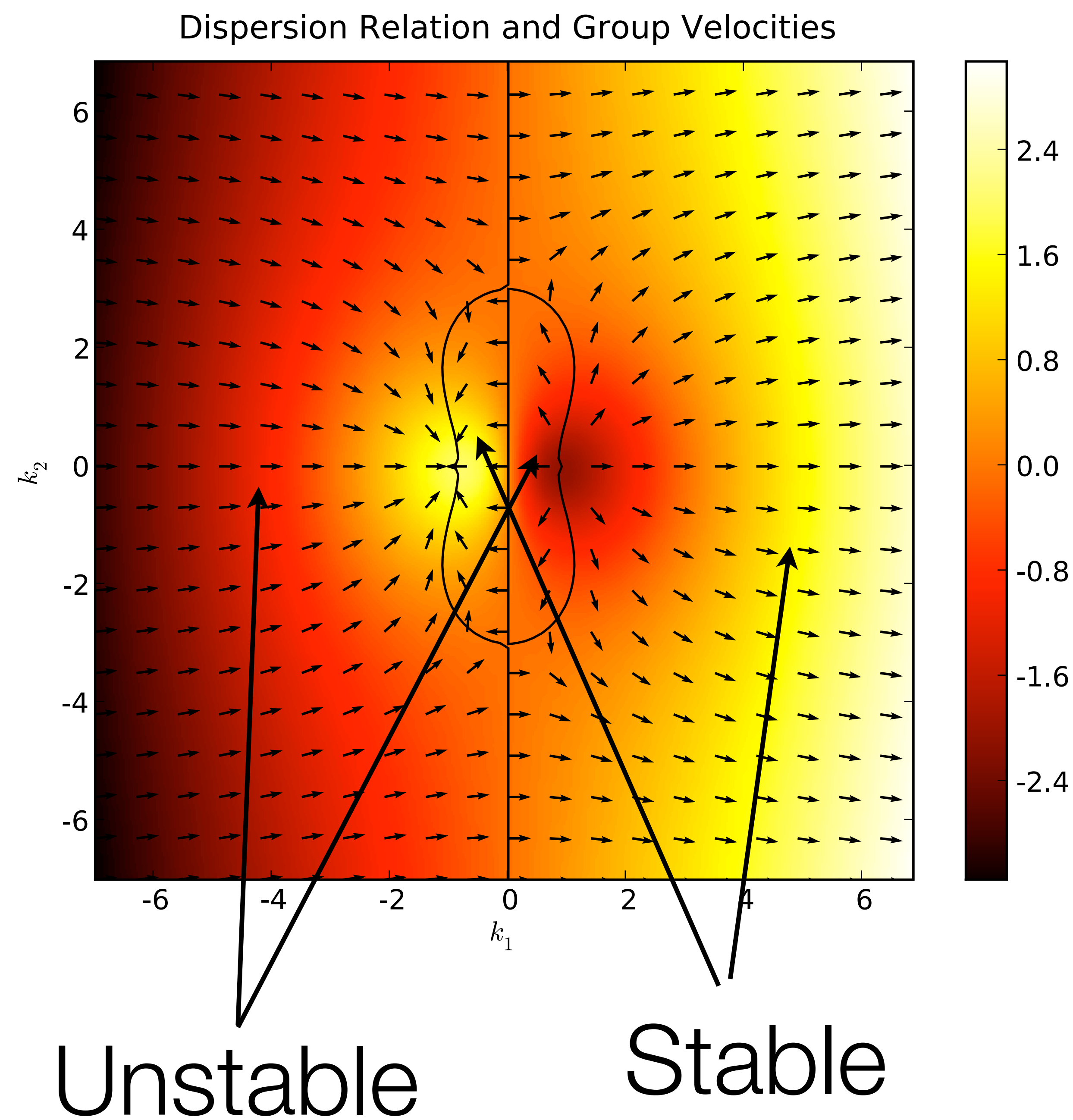


Dispersion Relations

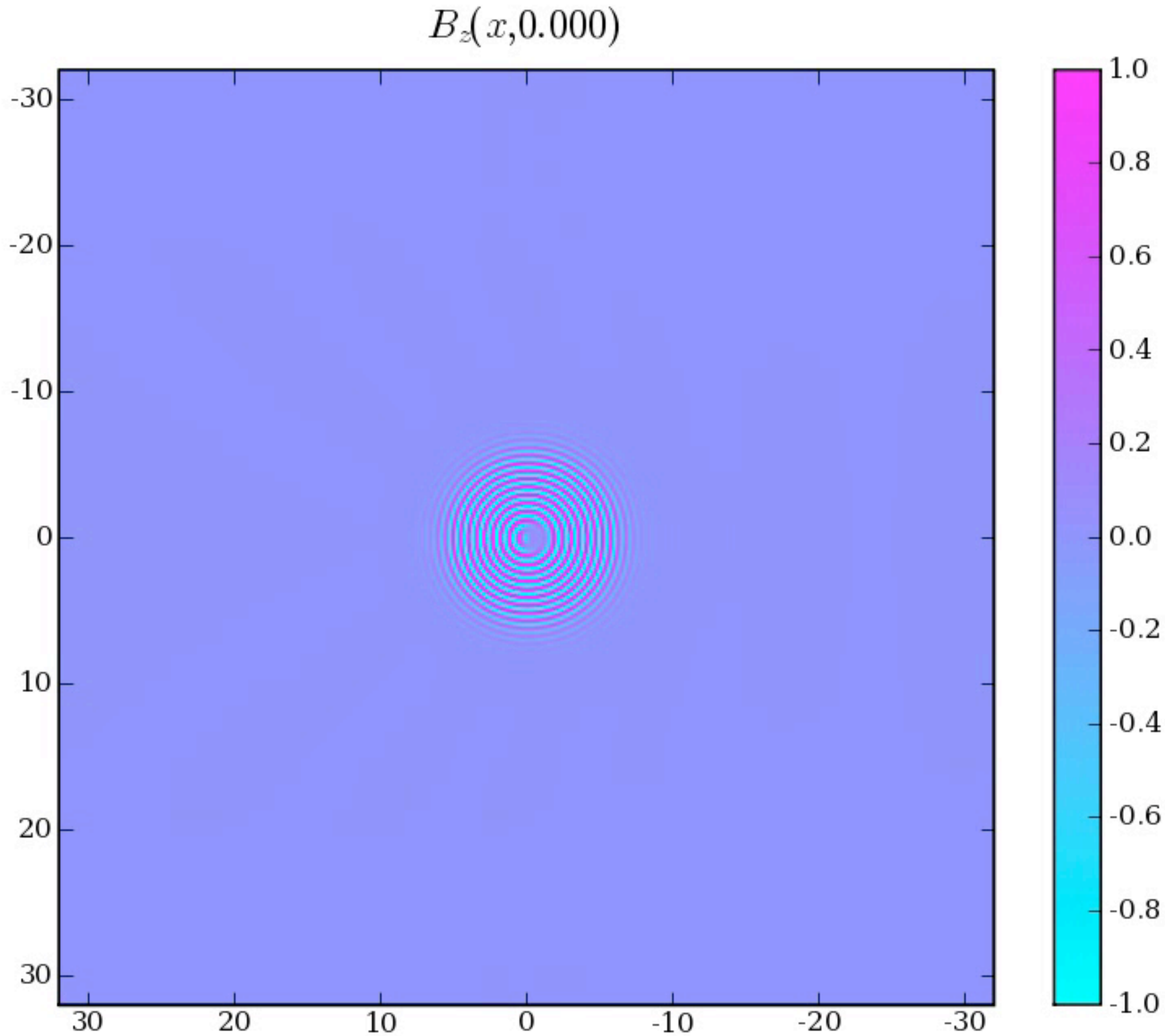
- Complicated dispersion relation, not quite hyperbolic
- PML unstable in y direction for

$$k_0 < 0$$

- PML unstable in x direction on irregular region

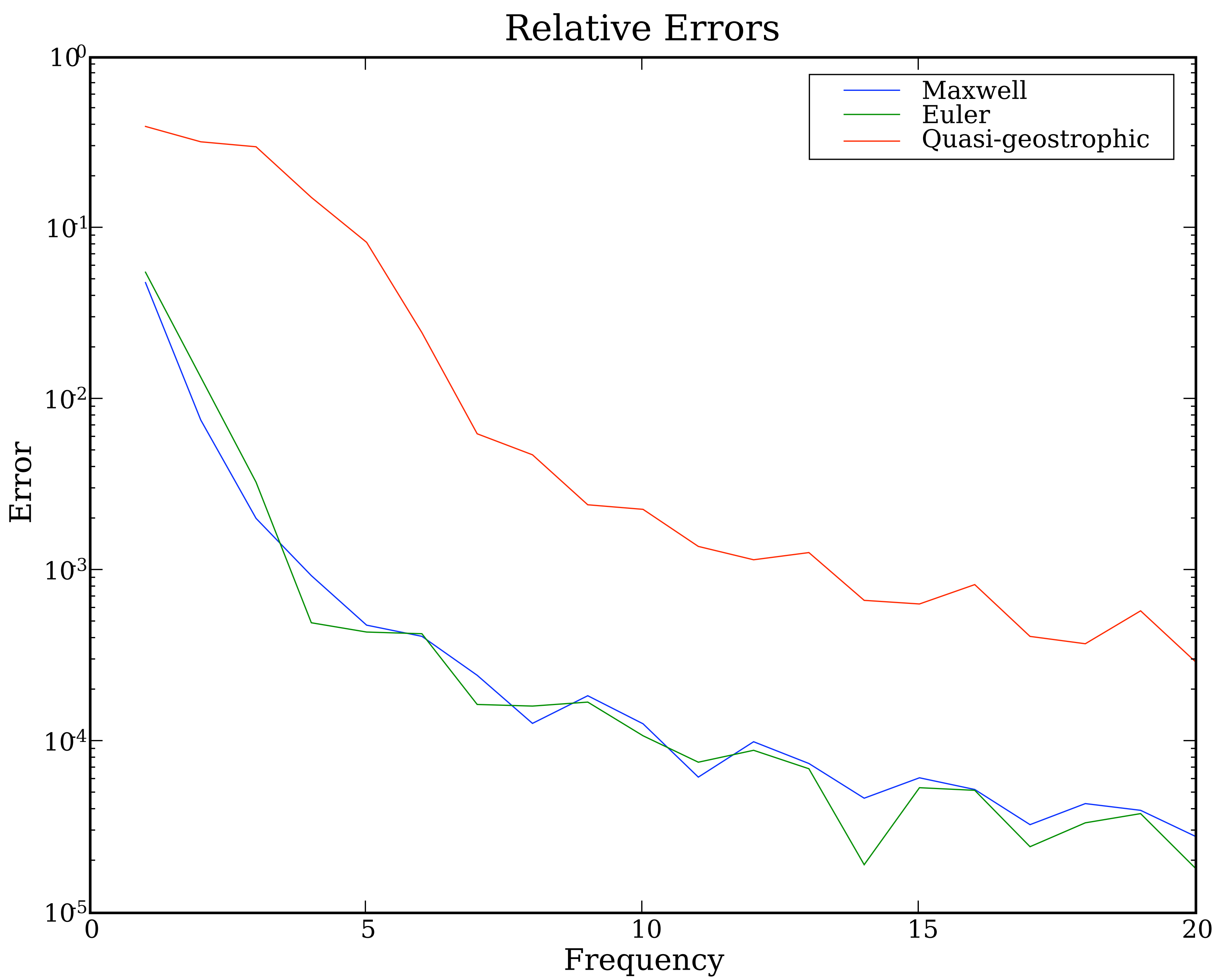


Tom Hagstrom can't fix this by completing the square.



Quasi-Geostrophic Equations

Errors



- Measured error as function of frequency of initial data.
- Errors are large for low frequencies, small for high.
- By increasing width of buffer, one reduce errors for low frequencies.

Higher error for quasi-geostrophic due to nonlocality. Can be fixed.

Stability

Stability of Phase Space Filtering

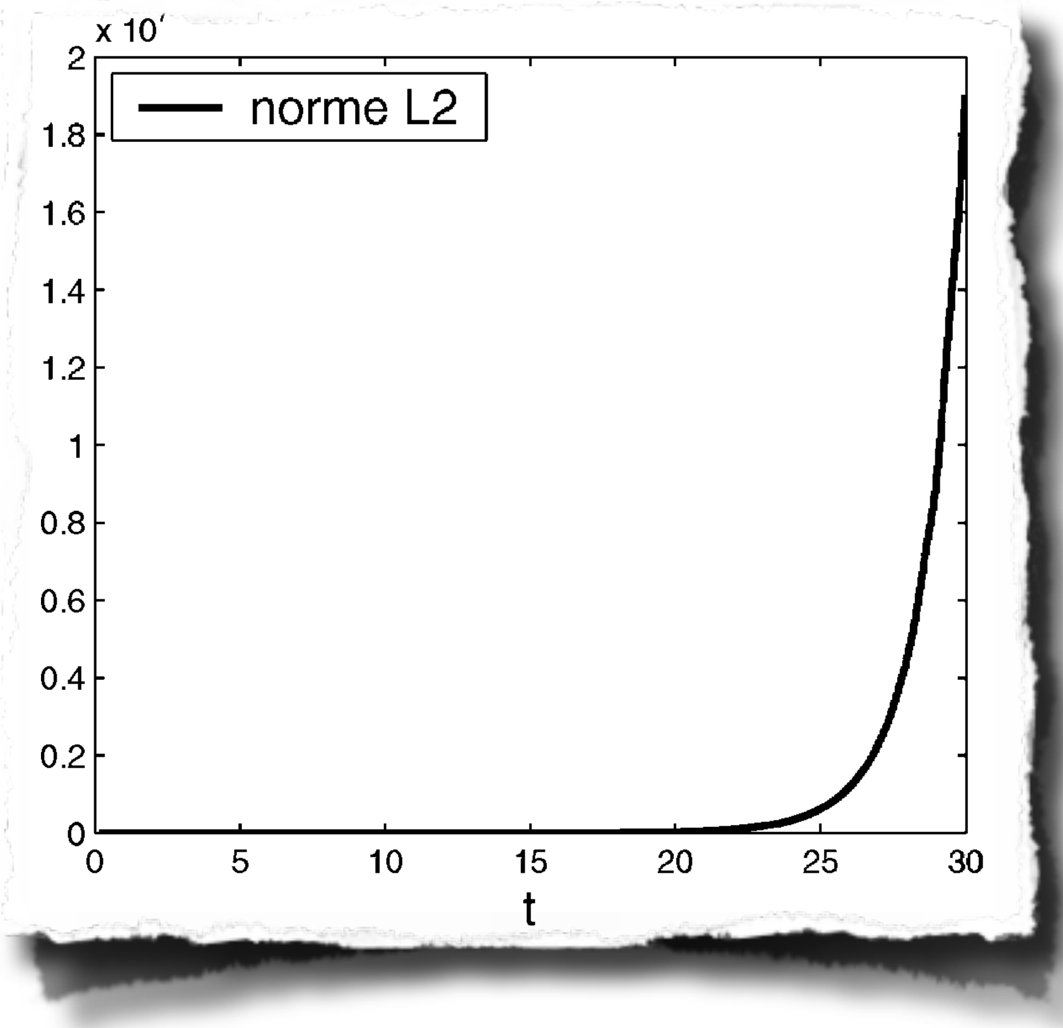
- Operator O^+ is self-adjoint, and $\sigma(O^+) \subseteq [0,1]$.
- Implies filtering is dissipative: $\sigma(1 - O^+) \subseteq 1 - [0,1] = [0,1]$
- Propagation operator has norm 1:

$$||e^{HT_s}\left[\prod_{\text{all sides}}(1 - O^+)\right]|| \leq ||e^{HT_s}|| \prod_{\text{all sides}} ||(1 - O^+)|| \leq 1 \prod_{\text{all sides}} 1 = 1$$

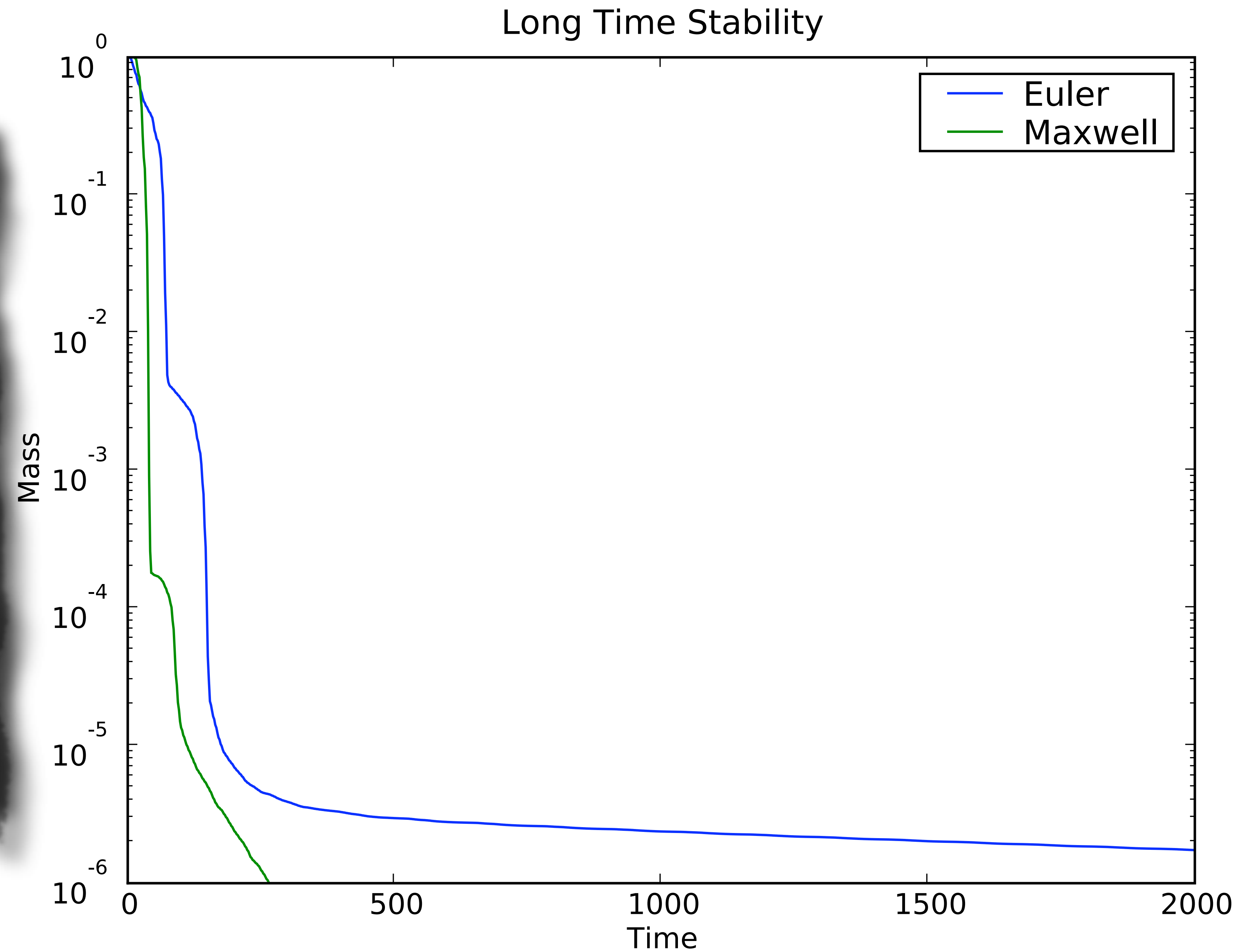
- Numerical solution is *strongly* stable:

$$||u(x,t)|| \leq ||u_0(x)||$$

PML



Phase Space Filter



Low Frequencies

The Low Frequency Problem

- Heisenberg Uncertainty principle limits phase space filters for low frequencies.

- Filter width $w = O(\ln(\epsilon)/k_{min})$:

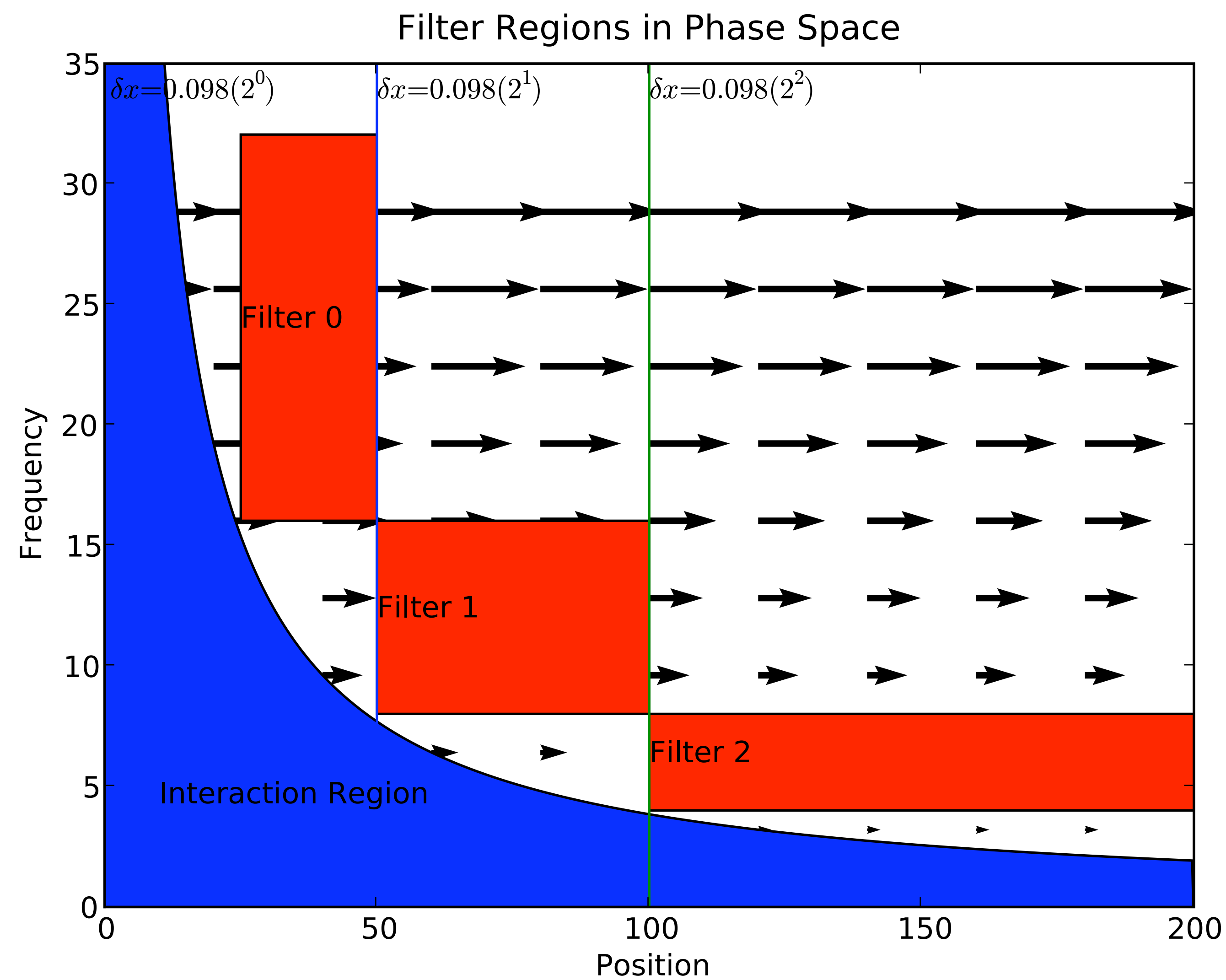
$$\text{Memory} = O((k_{max}/k_{min})^N)$$

- PML has similar issues: low frequencies dissipate over long distances.

- Dirichlet-to-Neumann immune to this problem in *homogeneous case*. In *inhomogeneous* case, Dirichlet-to-Neumann built using approximations valid only for high frequencies (Pseudo/Paradifferential calculus, see Szeftel).

Multiscale Solution

- Narrow filter for high frequency.
- Use filter with double the width to filter low frequencies; cut sampling rate in half.
- Filter width $w = O(\ln(\epsilon)/k_{min})$



$$\text{Memory} = O(\ln(\epsilon)\log_2(k_{max}/k_{min}))$$

The Low Frequency Problem: Resolution

- Implemented for 1-dimensional Schrodinger equation

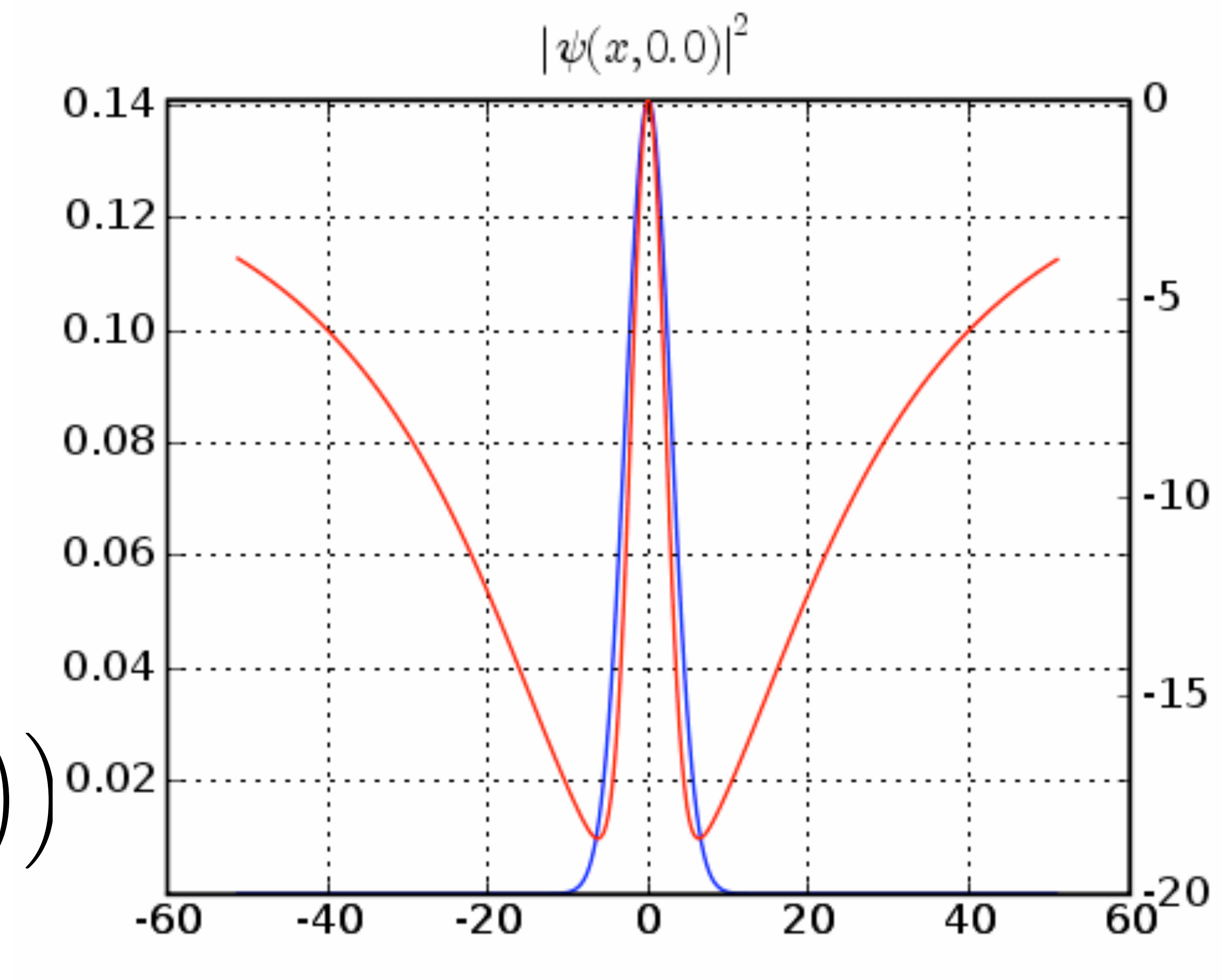
- Cost:

$$O(\log_2(k_{max}/k_{min}))$$

- If k_{min} is unknown, cost is:

$$O\left(T_{max}\log_2\left(T_{max}\frac{v_{k\approx 0}}{L}\right)\right)$$

- Works for long range potential/ inhomogeneity.



The Low Frequency Problem: Resolution

- Implemented for 1-dimensional Schrodinger equation

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- If k_{min} is unknown, cost is:

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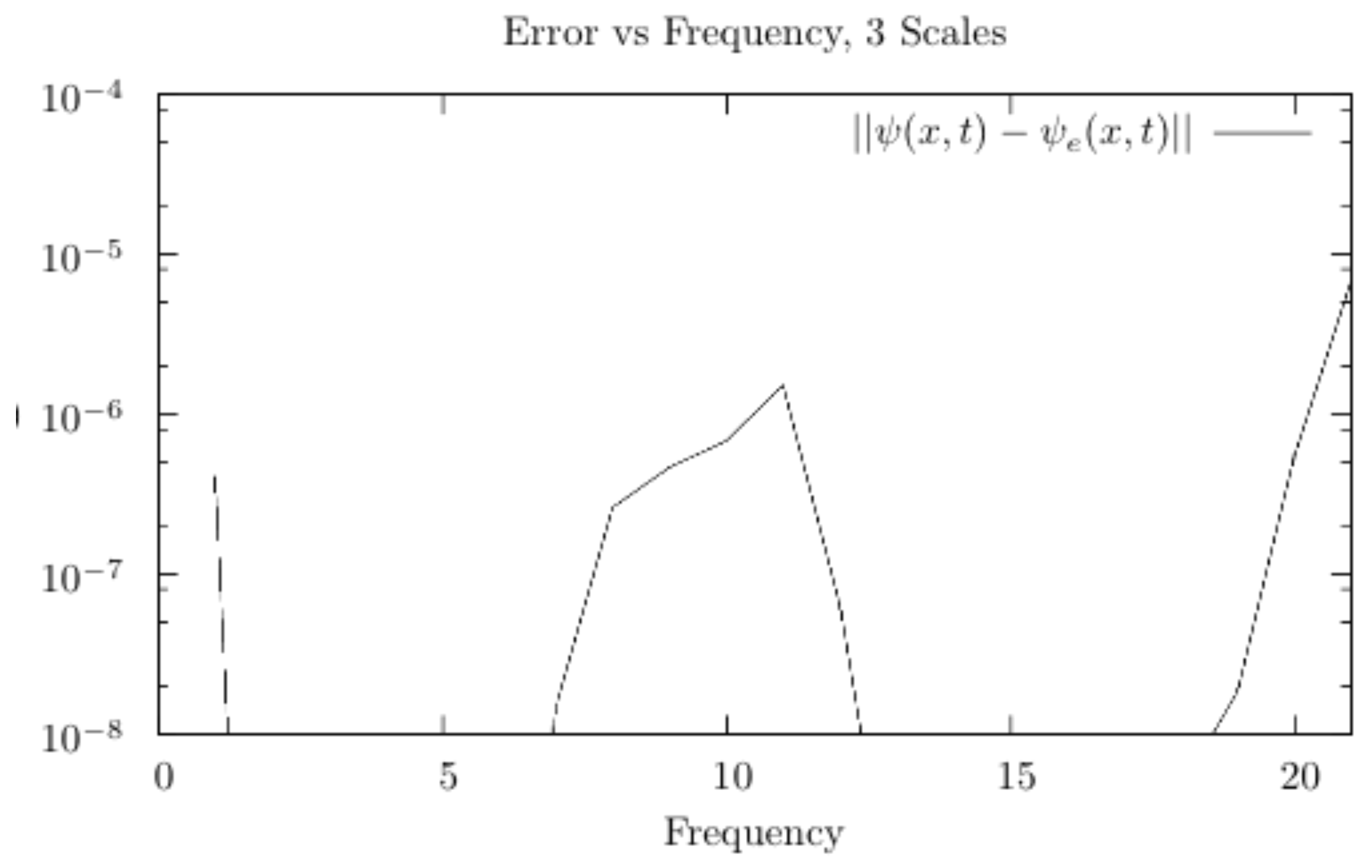

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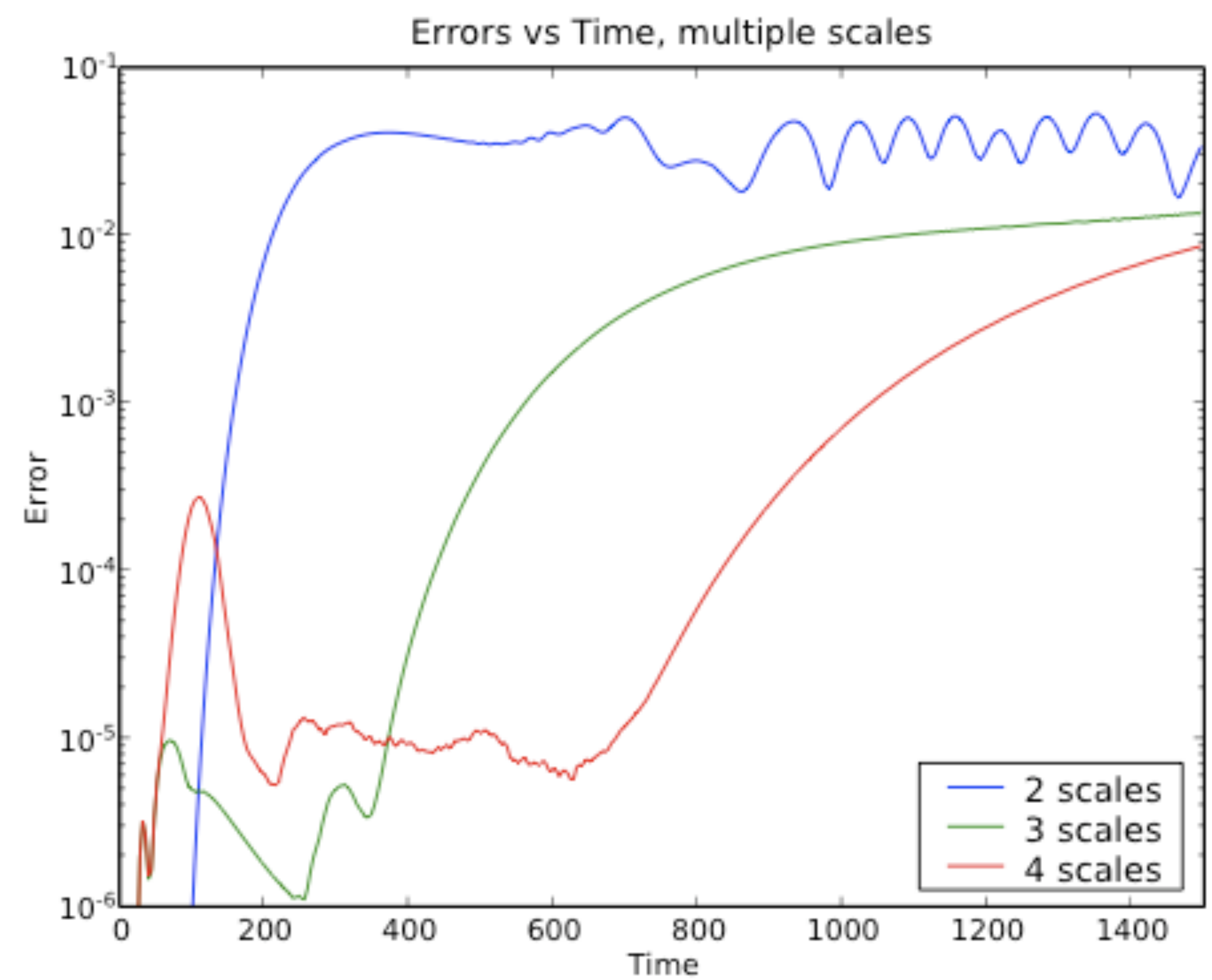

Conclusion

- Phase space filtering a new method of filtering outgoing waves.
- Works for anisotropic, inhomogeneous and even non-local waves.
- Stable and accurate: confirmed by rigorous theorem and numerical tests.

- [1] *Open Boundaries for the Nonlinear Schrodinger Equation*, with A. Soffer. JCP Vol. 225, Issue 2, p.p. 1218-1232. arXiv:math/0609183
- [2] *Multiscale Resolution of Shortwave-Longwave Interaction*, with A. Soffer. CPAM (accepted). arXiv:0705.3501
- [3] *Stable Open Boundaries for Anisotropic Waves*, with A. Soffer (submitted). arXiv:0805.2929

- All papers available from my webpage: http://cims.nyu.edu/~stucchio/