Phase Space Analysis in Medical Imaging

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Collaboration with L. Greengard.

Magnetic Resonance Imaging

- Excellent soft tissue contrast.
- No radiation.
- 2003 Nobel Prize (Lauterberger, Mansfield). Damadian maybe deserves credit too?

MY LATERAL SPINE

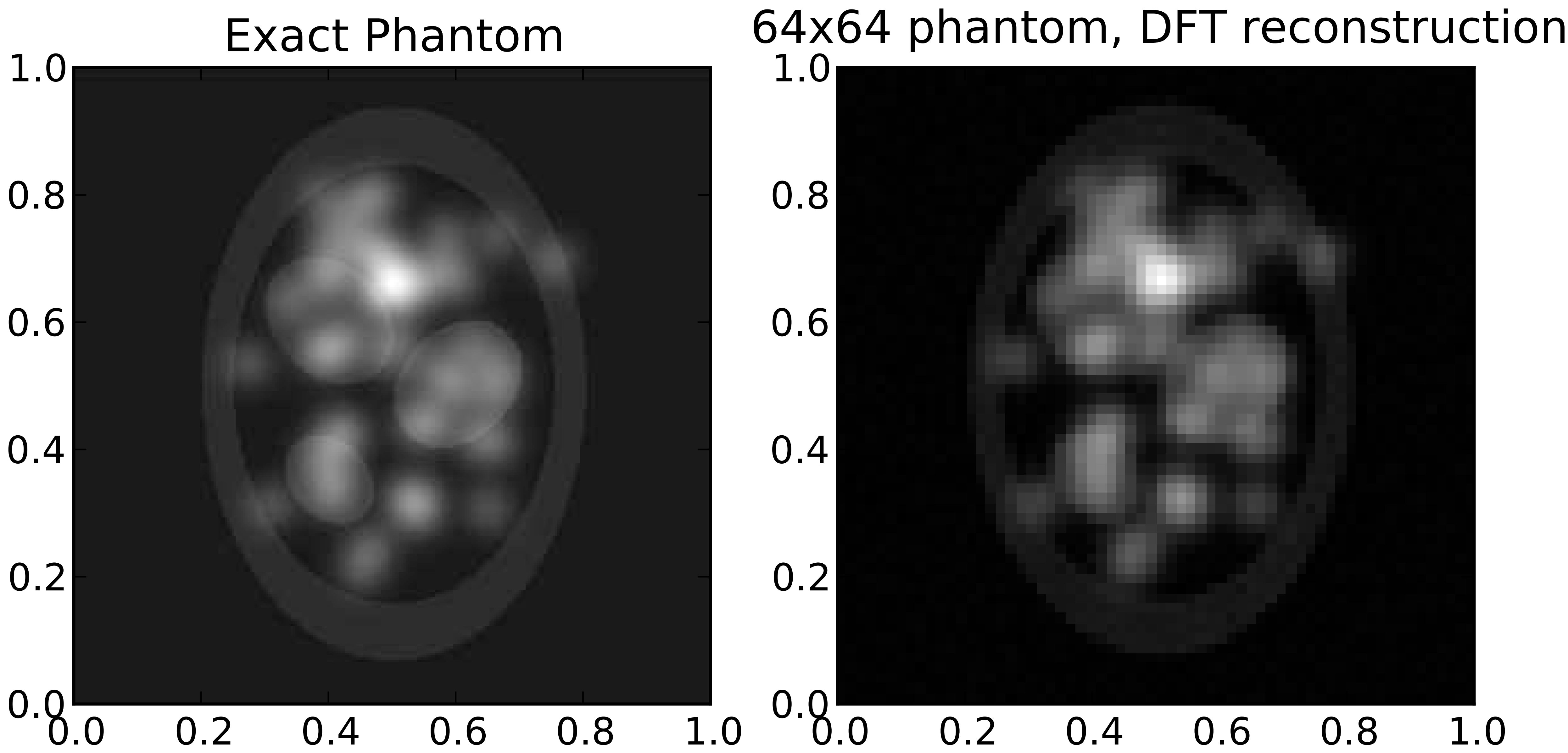
Objectives and Challenges

GOALS

- Show radiologist accurate pictures
- Quantify anatomical features

CHALLENGES

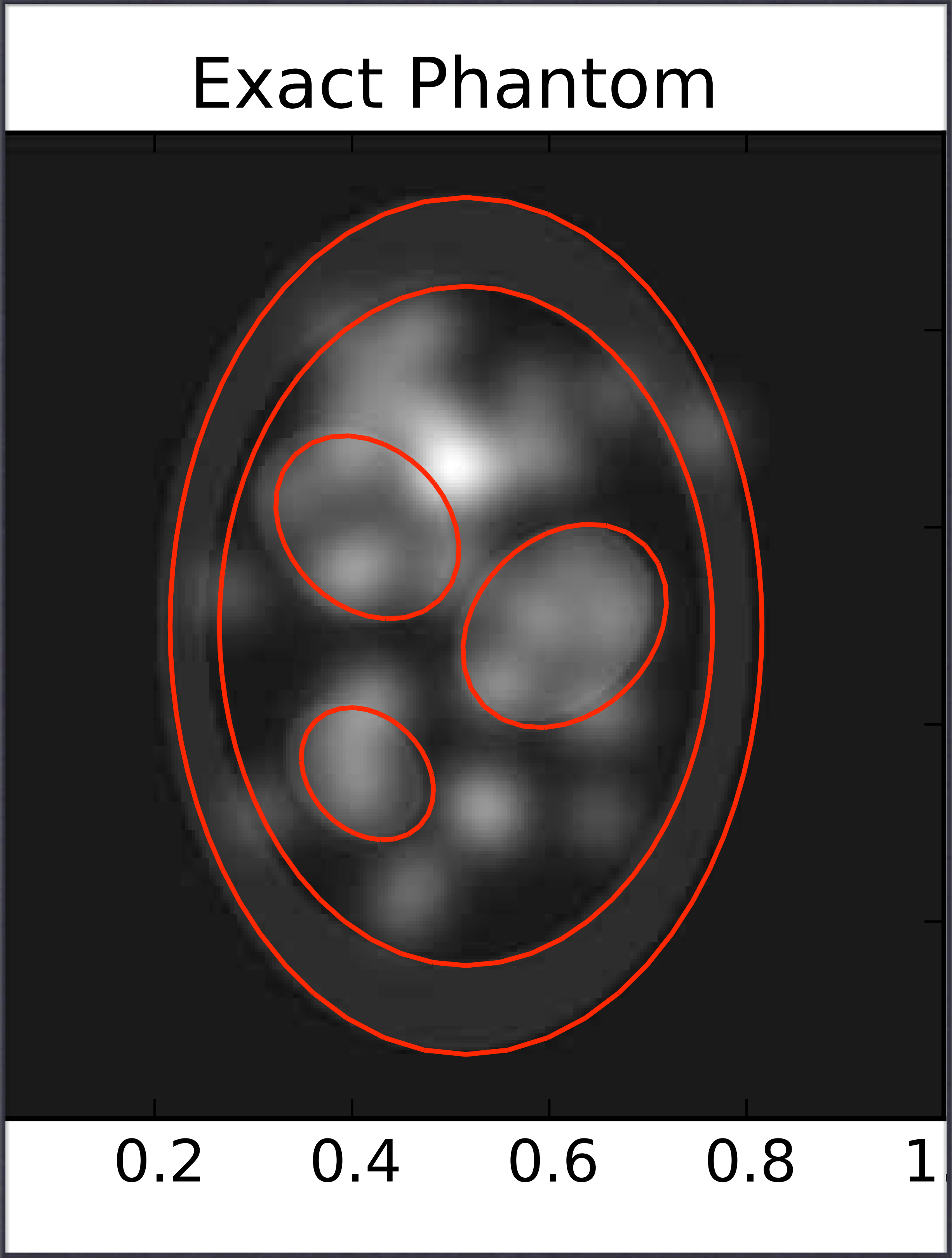
- Noise
- Artifacts
- Ambiguity



Accurate Pictures

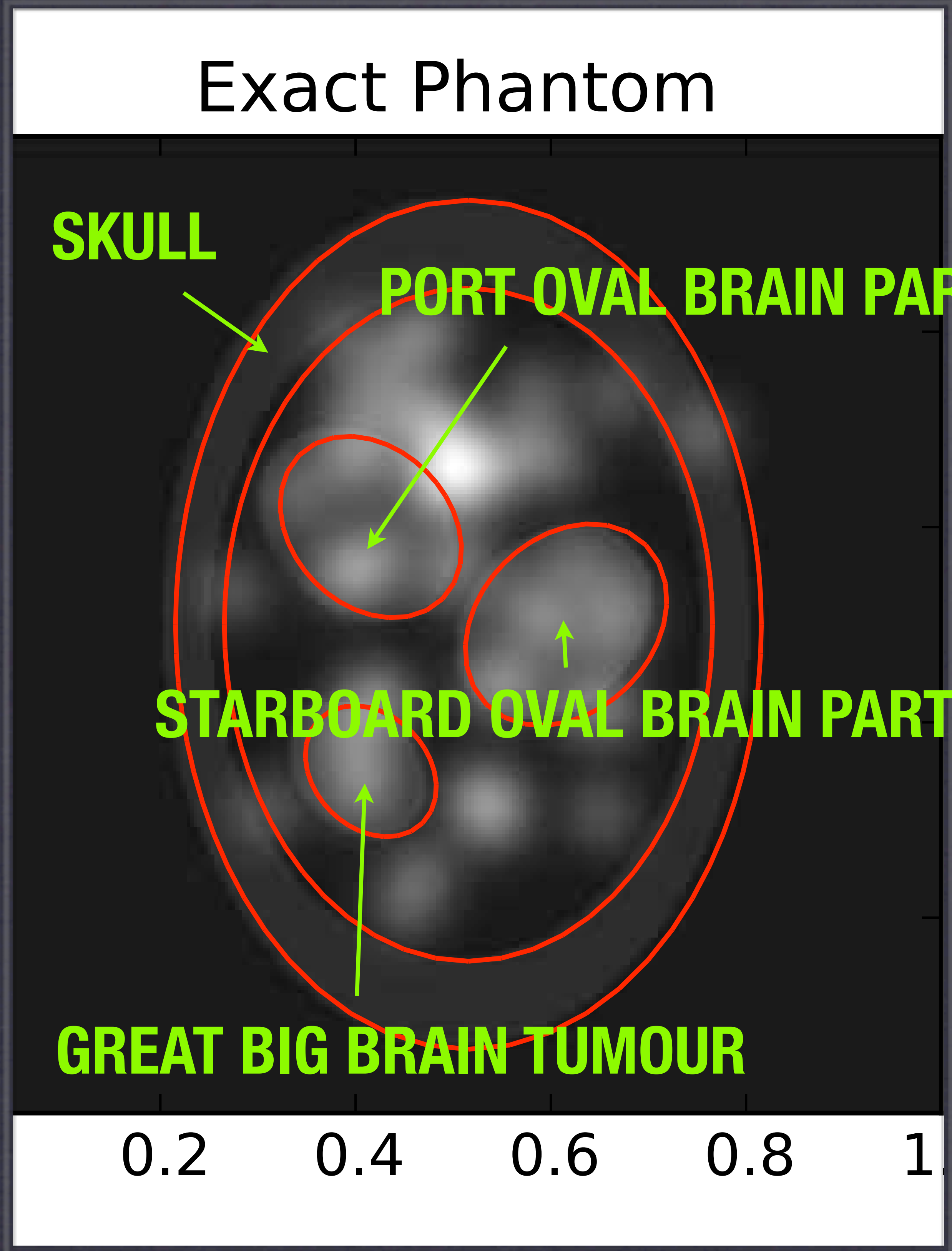
Segment Anatomical Features

Separate into distinct regions



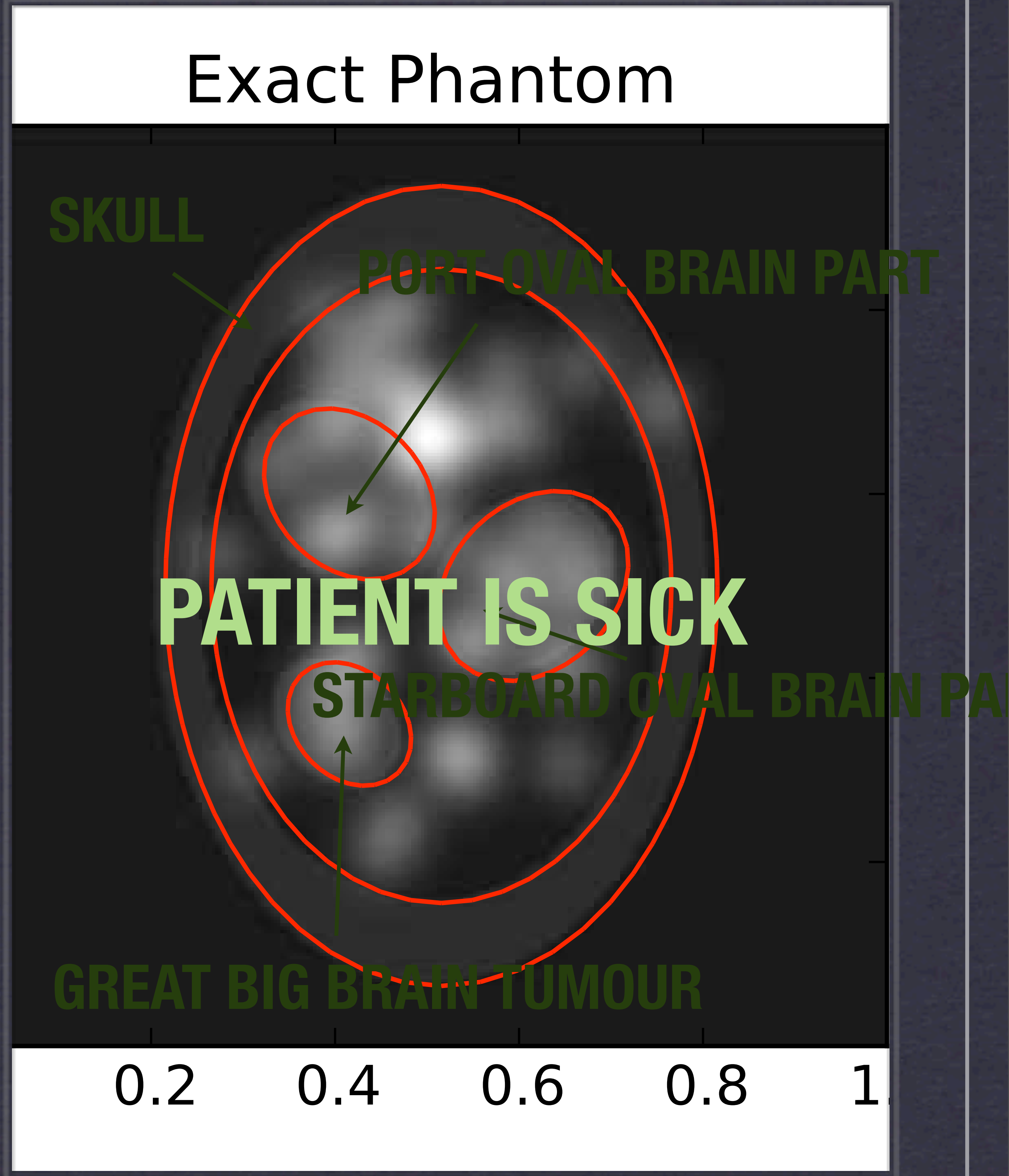
Identification

Label the segmented regions



Diagnosis

Draw conclusion from image data

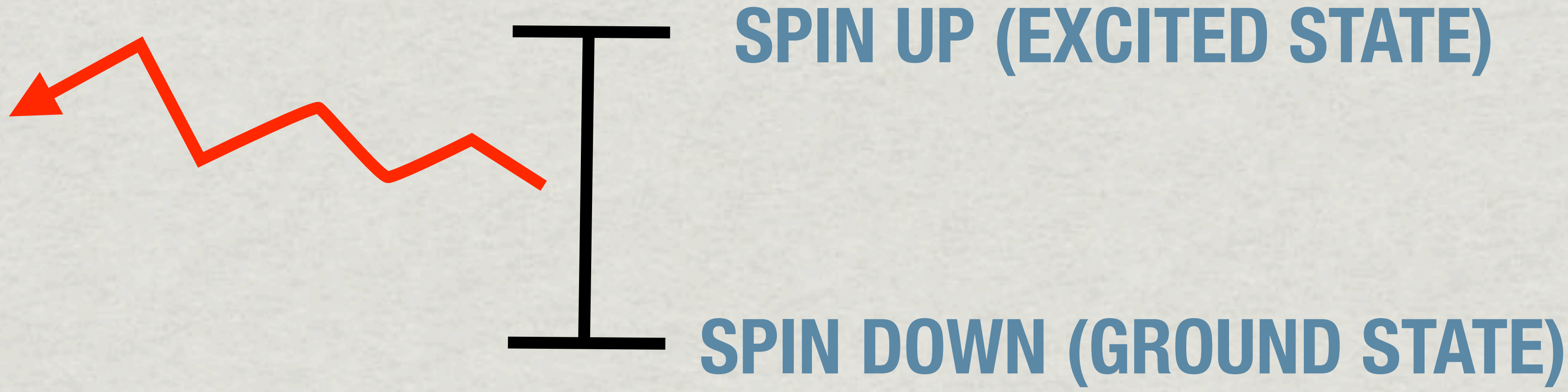


How an MRI works

How an MRI works

- Big Magnet: 1-2 Tesla
- Nucleus of atoms has spin
- Level Splitting: magnetic field breaks spin symmetry

How an MRI Works



- Excited state decays to ground, emits radiation.
- Measuring the radiation gives information on object.

How an MRI works

- Bloch Equation (macroscopic model):

$$\begin{aligned}
\partial_t \vec{M}(x,t) &= \gamma \vec{M} \times \vec{B}(x,t) - \\
&\quad \frac{P_{1,2}\vec{M}}{T_2} - \frac{P_3(\vec{M}(x,t) - M_0(x))}{T_1} \\
M_0(x) &= C\rho(x) \\
M(x,0) &= M_0(x)
\end{aligned}$$

- M(t) is magnetization, B(t) the magnetic field.

HOW AN MRI WORKS

COORDINATE SYSTEM

How an MRI works

- Hit system with weak RF pulse (excitation):

$$\vec{B}(x,t) = [0, f(t)w(x_z), 0]$$

$$\begin{aligned}\vec{M}(x,t) \times \vec{B}(x,t) &= [0, 0, M_0(x)] \times [0, f(t)w(x_3), 0] \\ &= [-M_0(x)f(t)w(x_3), 0, 0]\end{aligned}$$

- Rotates spins from z-direction into x-y plane

How an MRI works

- Switch off excitation pulse, use probe field:

$$\vec{B}(t) = [B_0 + \vec{G}(t) \cdot [x_1, x_2, 0]^T]\vec{z}$$

- X-Y components decoupled from Z component
- Substitution: $M(t) = \vec{M}_x(t) + i\vec{M}_y(t)$

How an MRI works

- Bloch Equation:

$$\partial_t M(x,t) = \left[-i\gamma(B_0 + G(t)\cdot x) - \frac{1}{T_2}\right] M(x,t)$$

$$M(x,t_0) = \rho(x) w(x_z) h(t_0)$$

How an MRI works

- Use RF receiver coils measure emission in the sample.

$$S(t) \sim \int M(x,t)dx + \text{noise}$$

How an MRI works

- Solution:

$$M(x,t) = \rho(x) w(x_3) e^{-i\gamma B_0 t} e^{-i\gamma (\int_{t_0}^{t} G(t')dt')\cdot x} e^{-t/T_2}$$

- Simplify:

$$\begin{aligned} \vec{k}(t) &= \gamma \int_{t_0}^{t} G(t')dt' \\ M(x,t) &\mapsto e^{i\gamma B_0 t} M(x,t) \end{aligned}$$

How an MRI works

- Solution:

$$M(x,t) = \rho(x) w(x_3) e^{-ik(t)\cdot x} e^{-t/T_2}$$

- Signal:

$$S(t) \sim e^{-t/T_2} \int \rho(x) w(x_3) e^{-ik(t)\cdot x} dx + \text{noise}$$

How an MRI works

- Signal:

$$S(t) \sim e^{-t/T_2}\hat{\rho}(k(t)) + \text{noise}$$

- An MRI measures the **Continuous** Fourier Transform of the density.

Image Reconstruction

ACCURATE PICTURES

Fourier Inversion

- Hugely ill posed problem.

$$\text{Given } \hat{\rho}(k_1), \ldots, \hat{\rho}(k_N), \text{ find } \rho(x)$$

- Then:

$$\exists f(x) \neq 0, \widehat{[\rho + f]}(k_{1,\ldots,N}) = \hat{\rho}(k_{1,\ldots,N})$$

Fourier Inversion

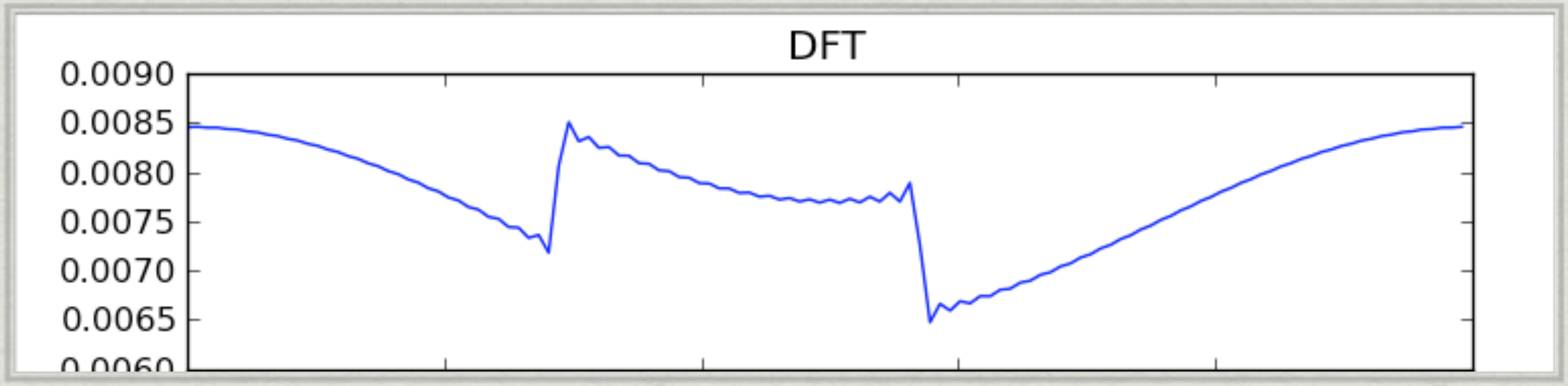
- Fourier's Theorem. Assume Cartesian sampling.

$$\rho(x) \approx \sum_{\vec{n}} \hat{\rho}(2\pi\vec{n})e^{-i2\pi\vec{n}x}$$

- Best approximation to density in $L^2([0,1]^2)$ norm

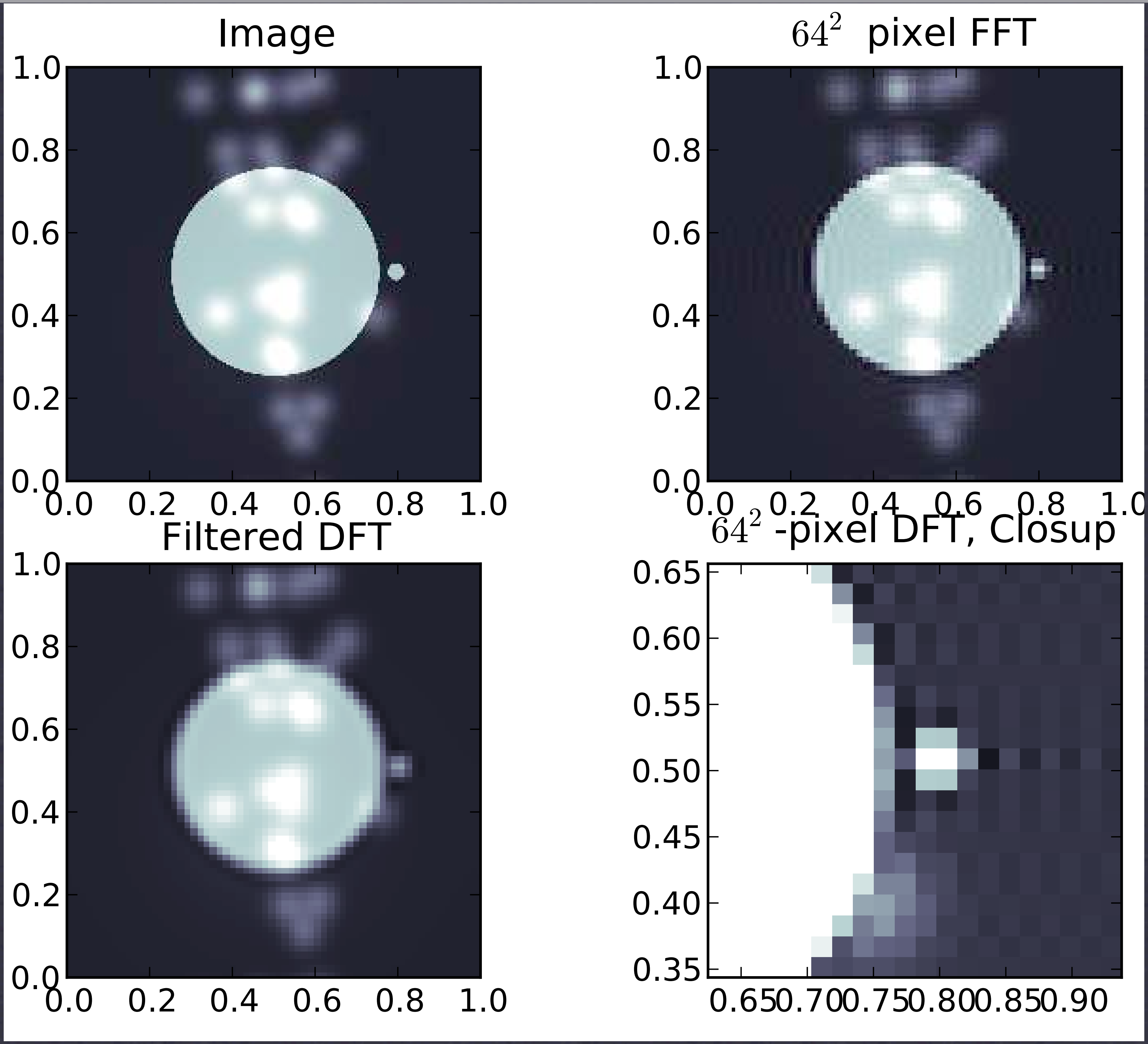
Fourier Inversion

- Fourier Transform not convergent pointwise

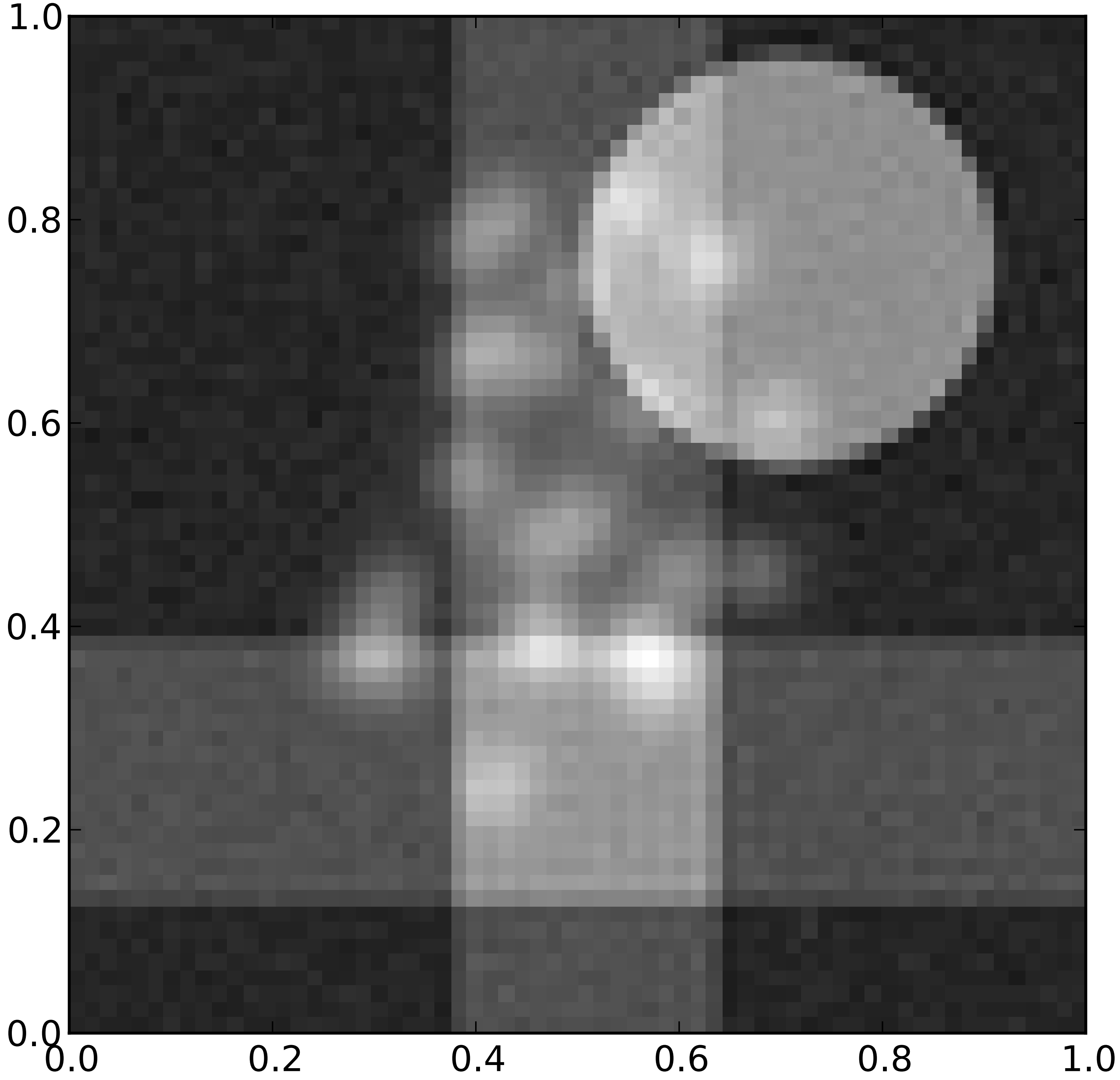


- Regularization discards information

$$\rho(x) \approx \sum_{\vec{n}} \hat{\rho}(2\pi\vec{n})w(\vec{n})e^{-i2\pi\vec{n}x}$$



FOURIER INVERSION



OTHER ARTIFACTS

SMALL CURVATURE POSES PROBLEMS

Current Solution

- Reconstruct image using regularized discrete Fourier transform:

$$\rho(x) \approx \sum_{\vec{n}} \hat{\rho}(2\pi\vec{n})w(\vec{n})e^{-i2\pi\vec{n}x}$$

- Clean up regularized image in x-domain.
- Segment/identify based on cleaned up image.

Segmentation

OUTLINING THE IMPORTANT FEATURES

Segmentation

GOALS

- Segmentation by anatomy/composition - outline the cancerous part
- Segmentation by perception - draw the same outlines as a human
- Image-space segmentation - separate based on image boundaries

Image boundaries

- Image boundaries are places where image composition changes sharply.
- In medical images, this happens at discontinuities of image.
- Not true in other modalities.

Discontinuities

- Want to find discontinuities of an image.
- Image domain methods fail due to artifacts.
- Want to find discontinuities from raw MRI data, i.e. from samples of Fourier transform of image.

Discontinuities

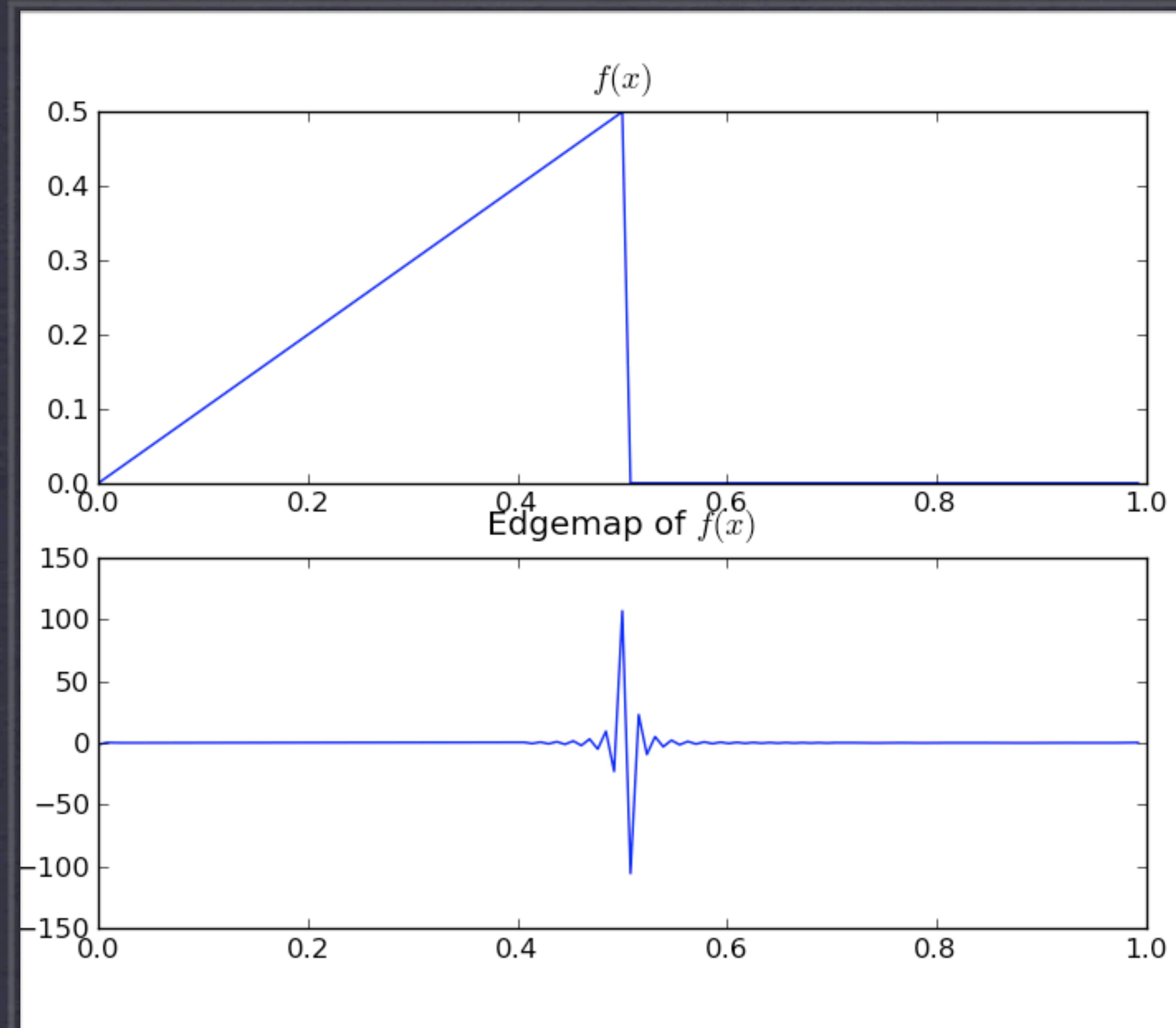
- Simple model: a 1-d function with a discontinuity:

$$\int e^{ikx} f(x)dx = e^{ikx_0} \frac{f(x_0^+) - f(x_0^-)}{ik} + O(k^{-2})$$

- If we localize on high frequencies, we can extract edges.

1D Edge Detection

Laplace Filters, Gradient Filters, Concentration Kernels, etc.



STATE OF THE ART:
CONCENTRATION KERNELS, C.F. TADMOR/GELB/ETC

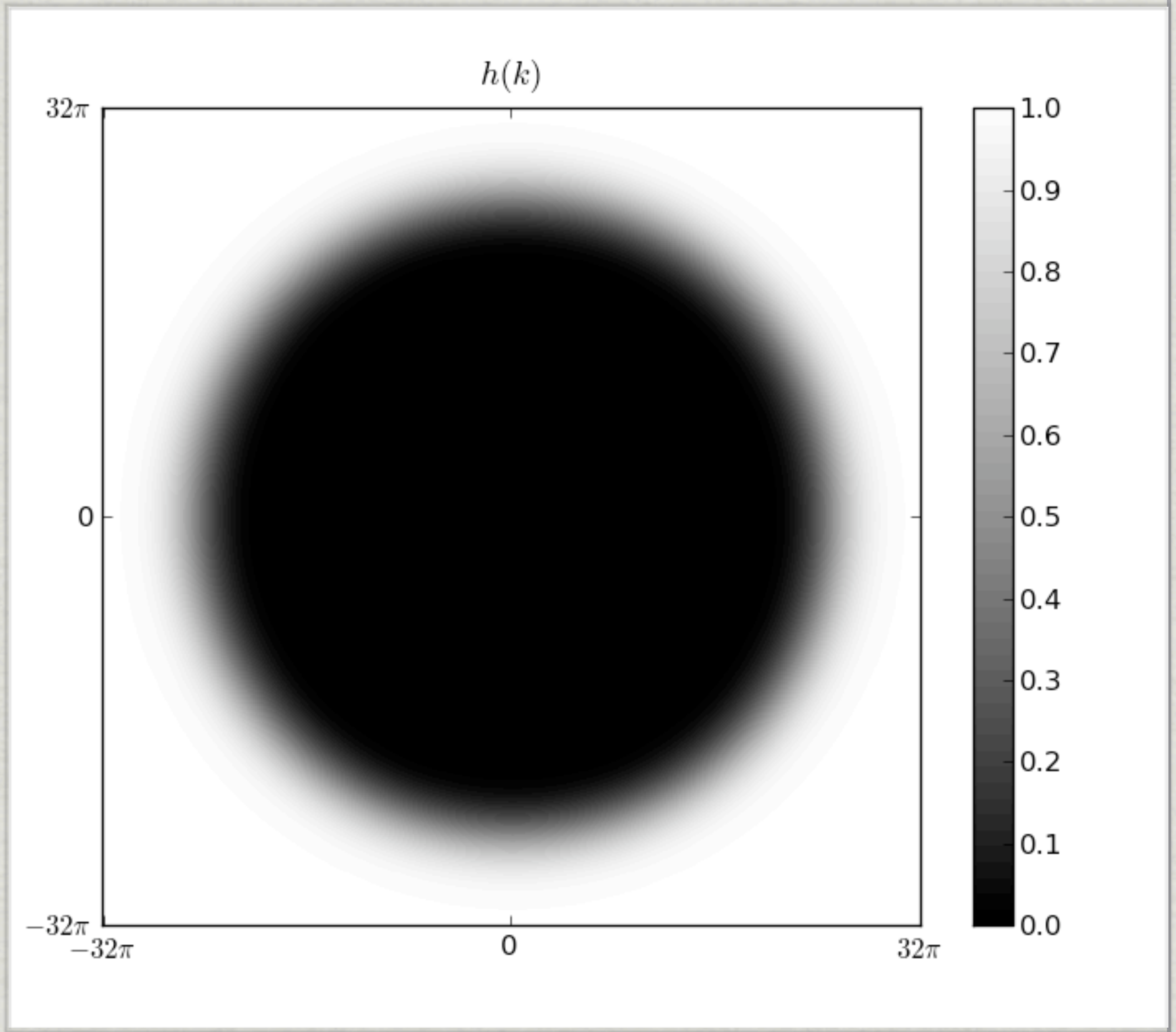
2D Edge Detectors

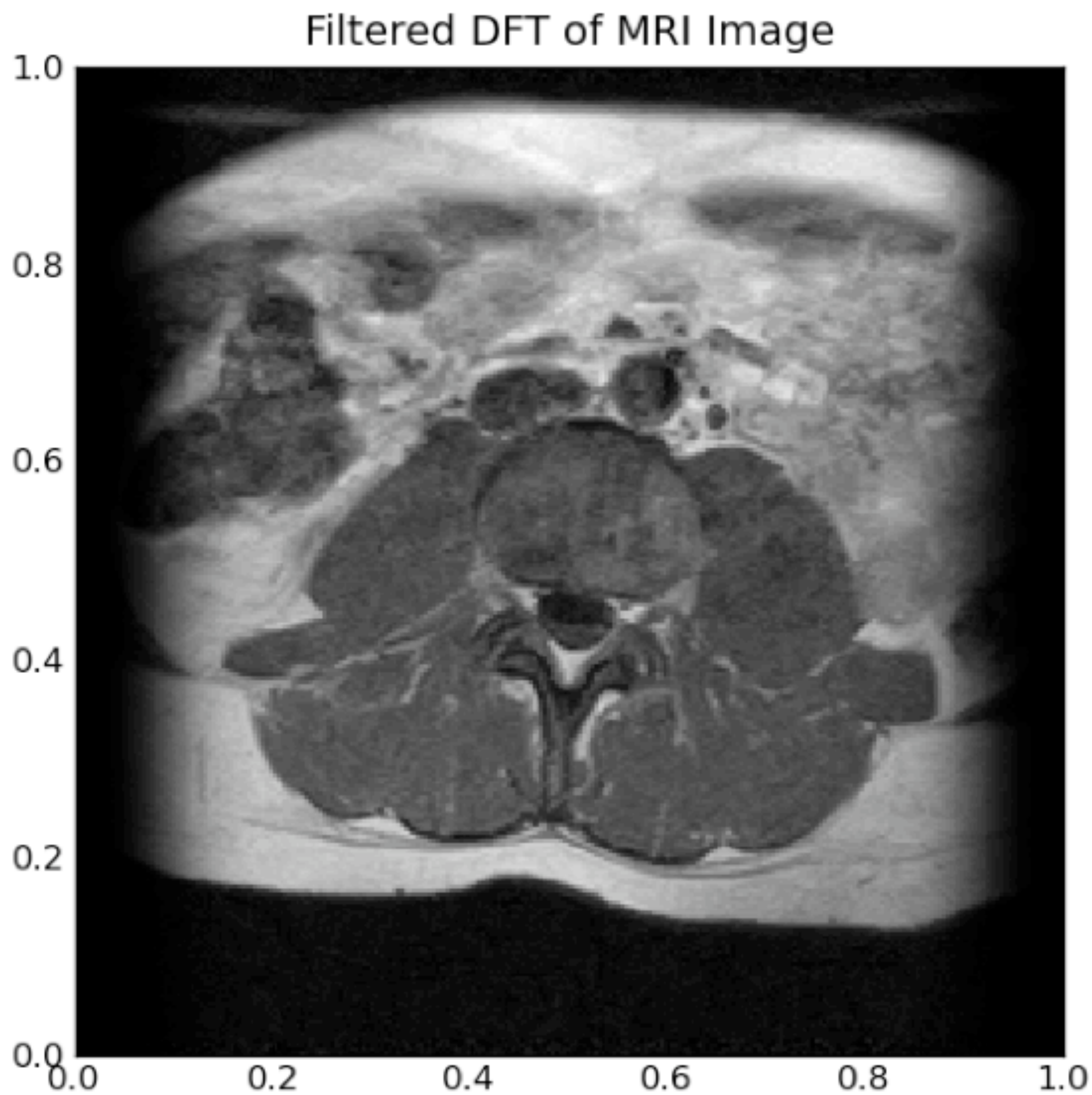
- Tensor Products

$$\mathbb{R}^2 = \mathbb{R} \otimes \mathbb{R}$$

- Radial Variables

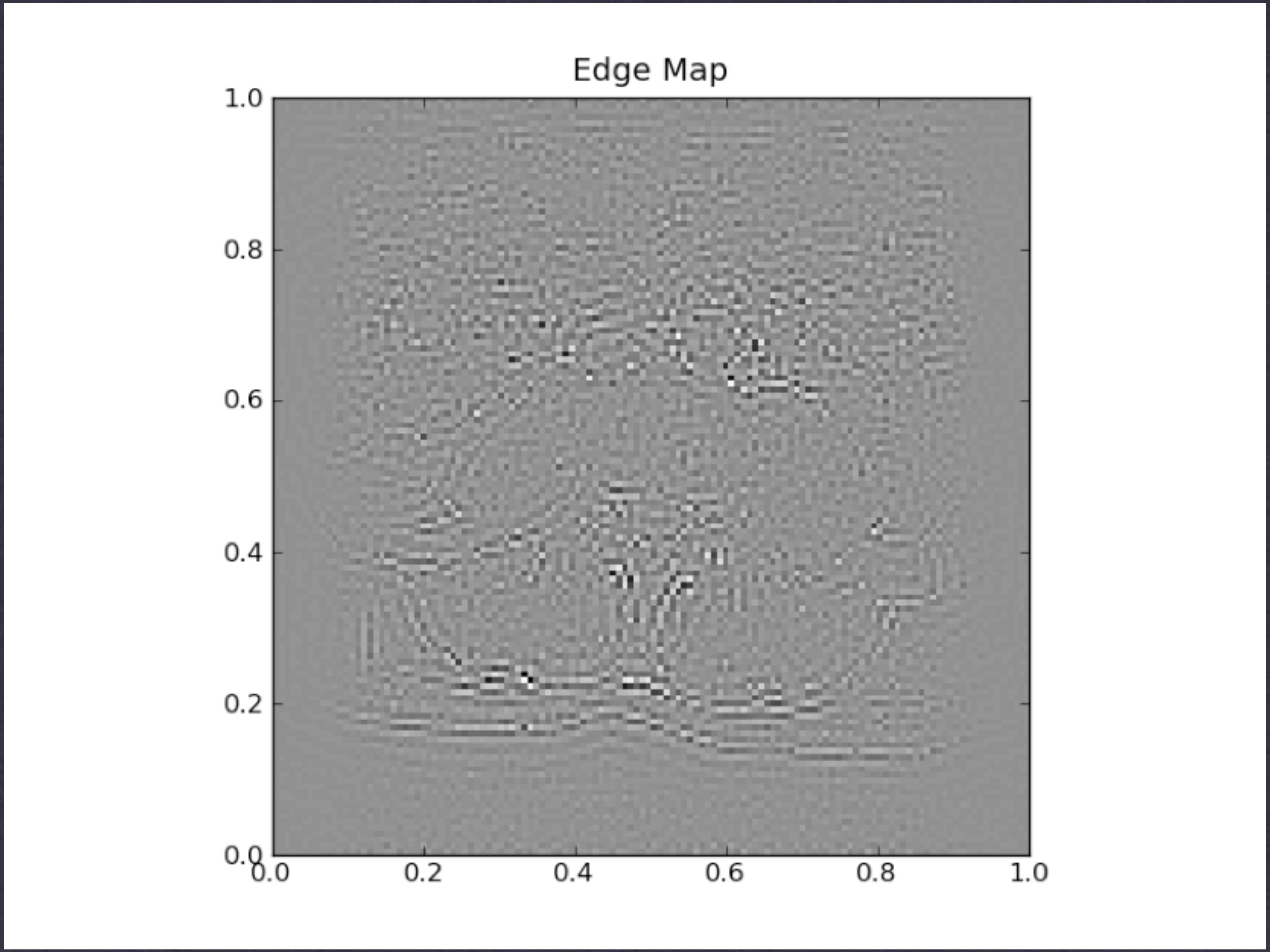
$$\mathrm{DFT}^{-1}[h(\vec{k})\hat{\rho}(\vec{k})]$$





RESULT OF HIGH FREQUENCY FILTERS

EDGE DETECTOR RESOLUTION IS HALF THAT OF IMAGE



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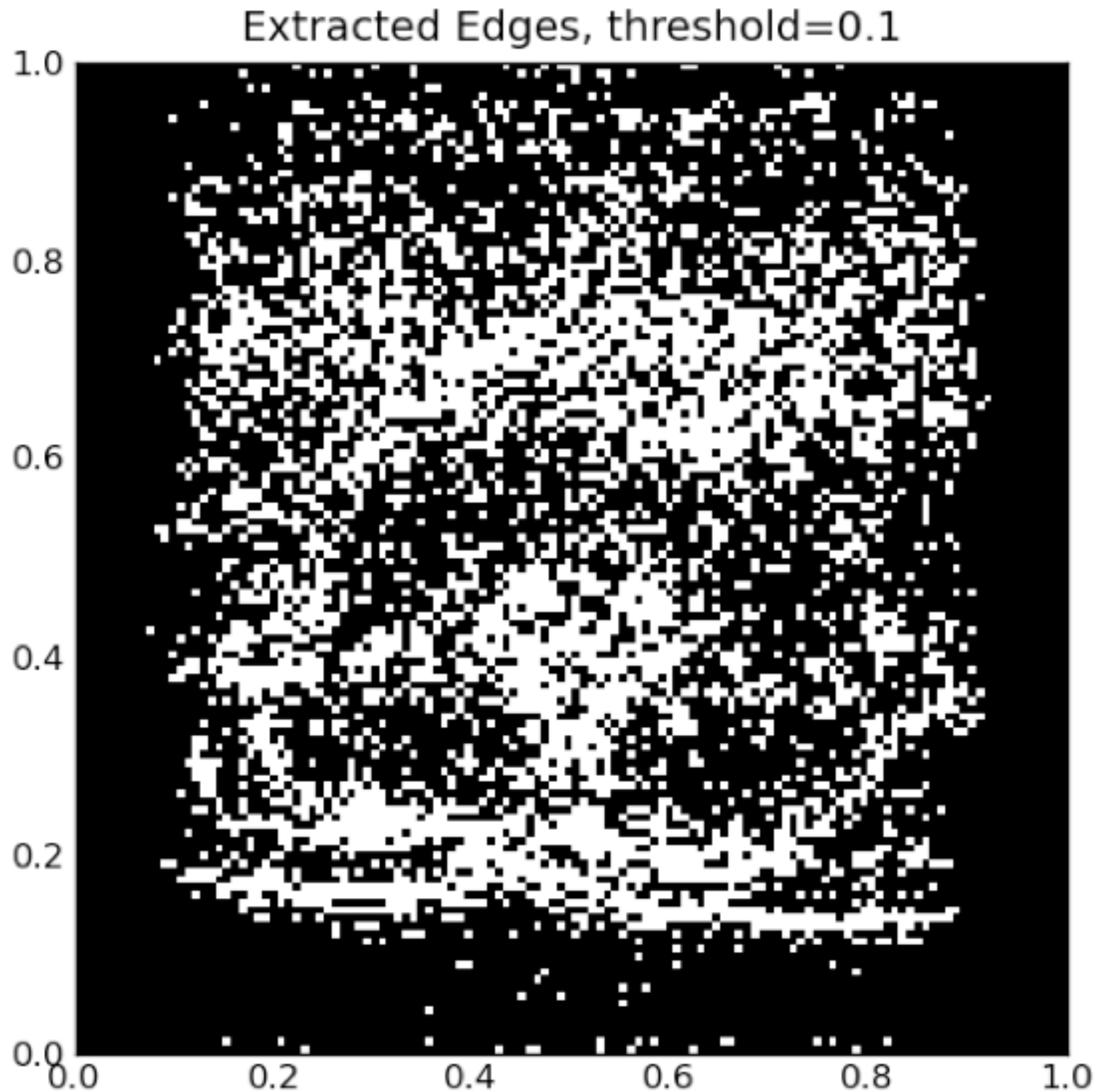

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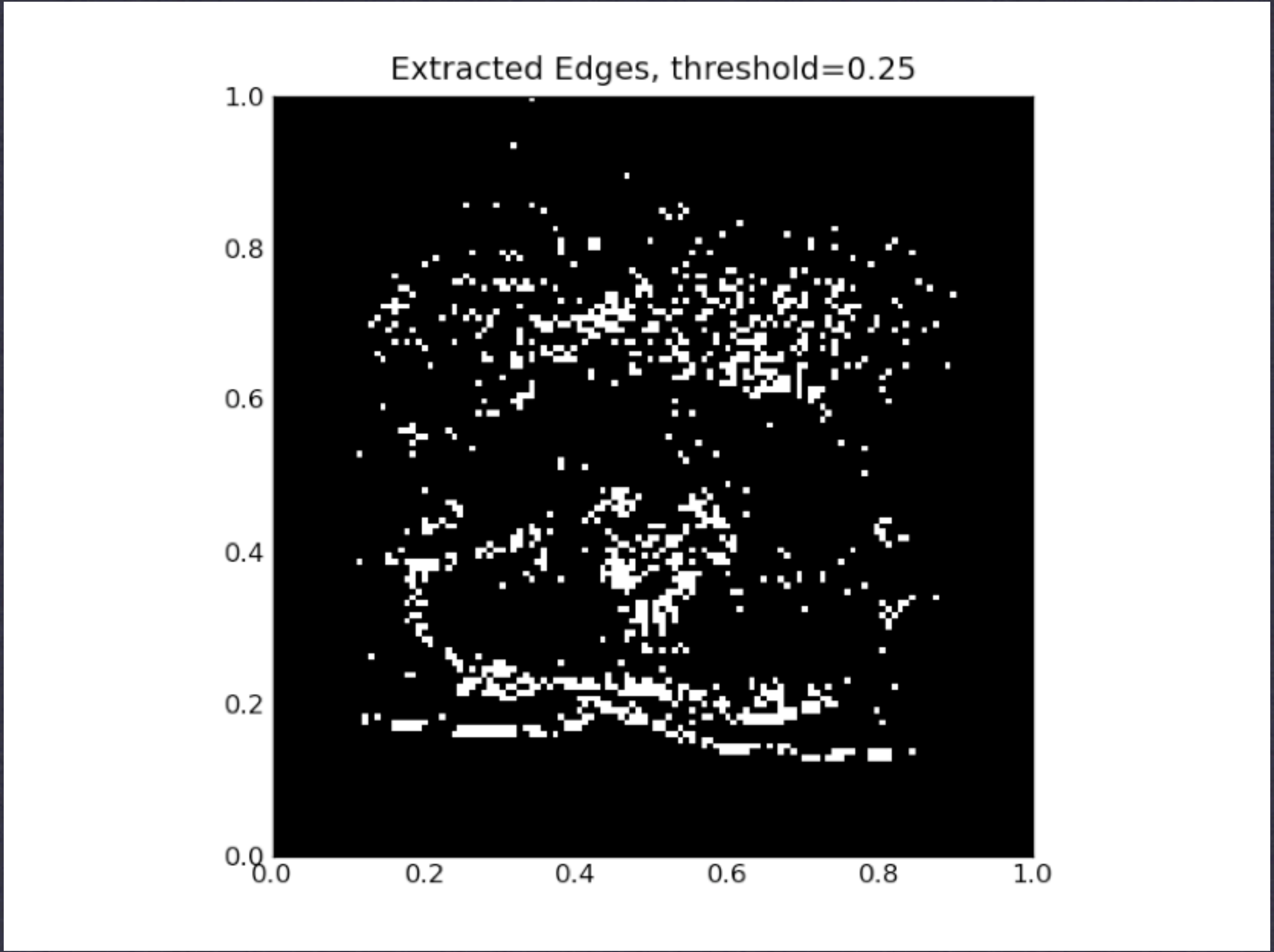

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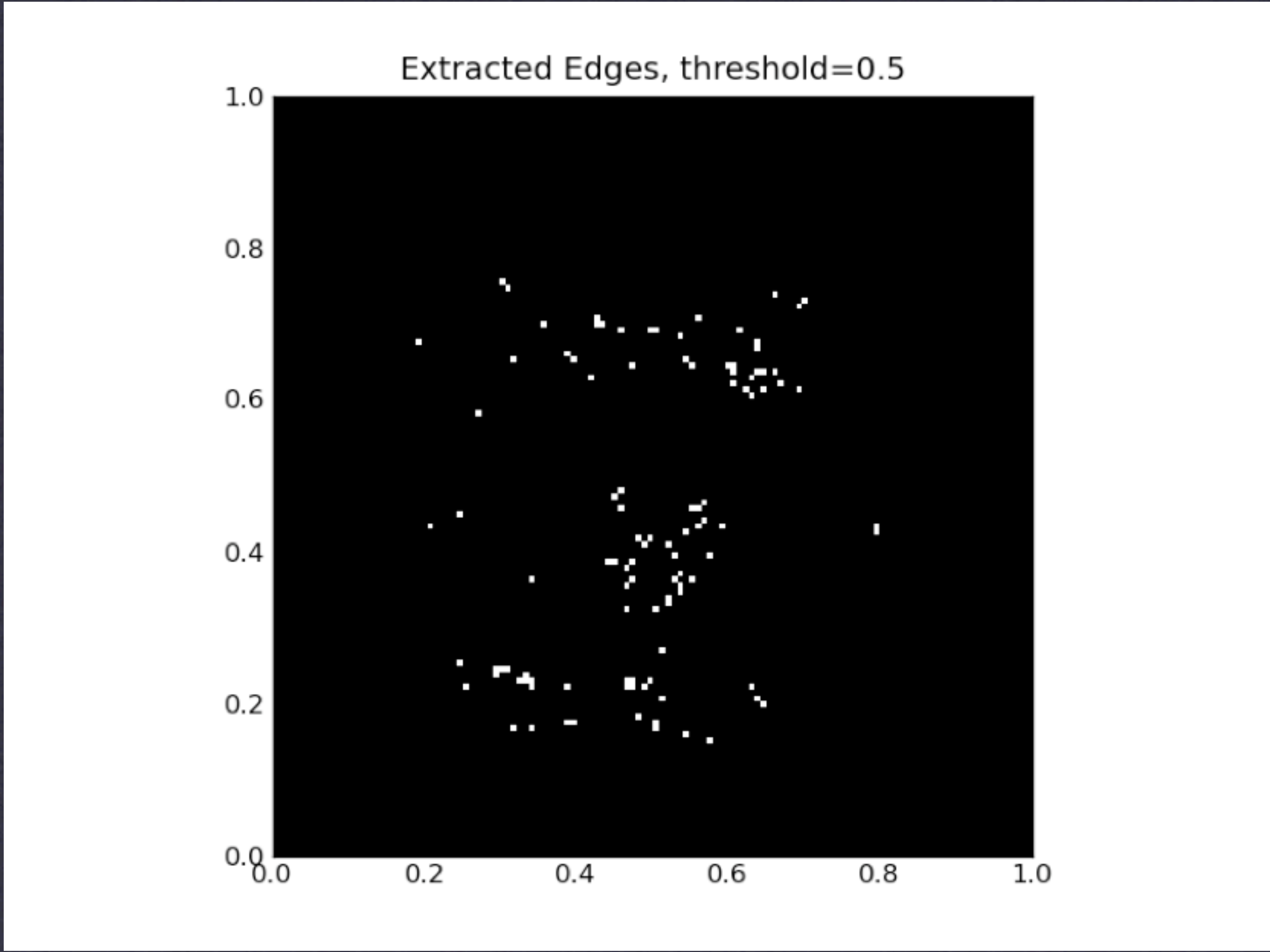

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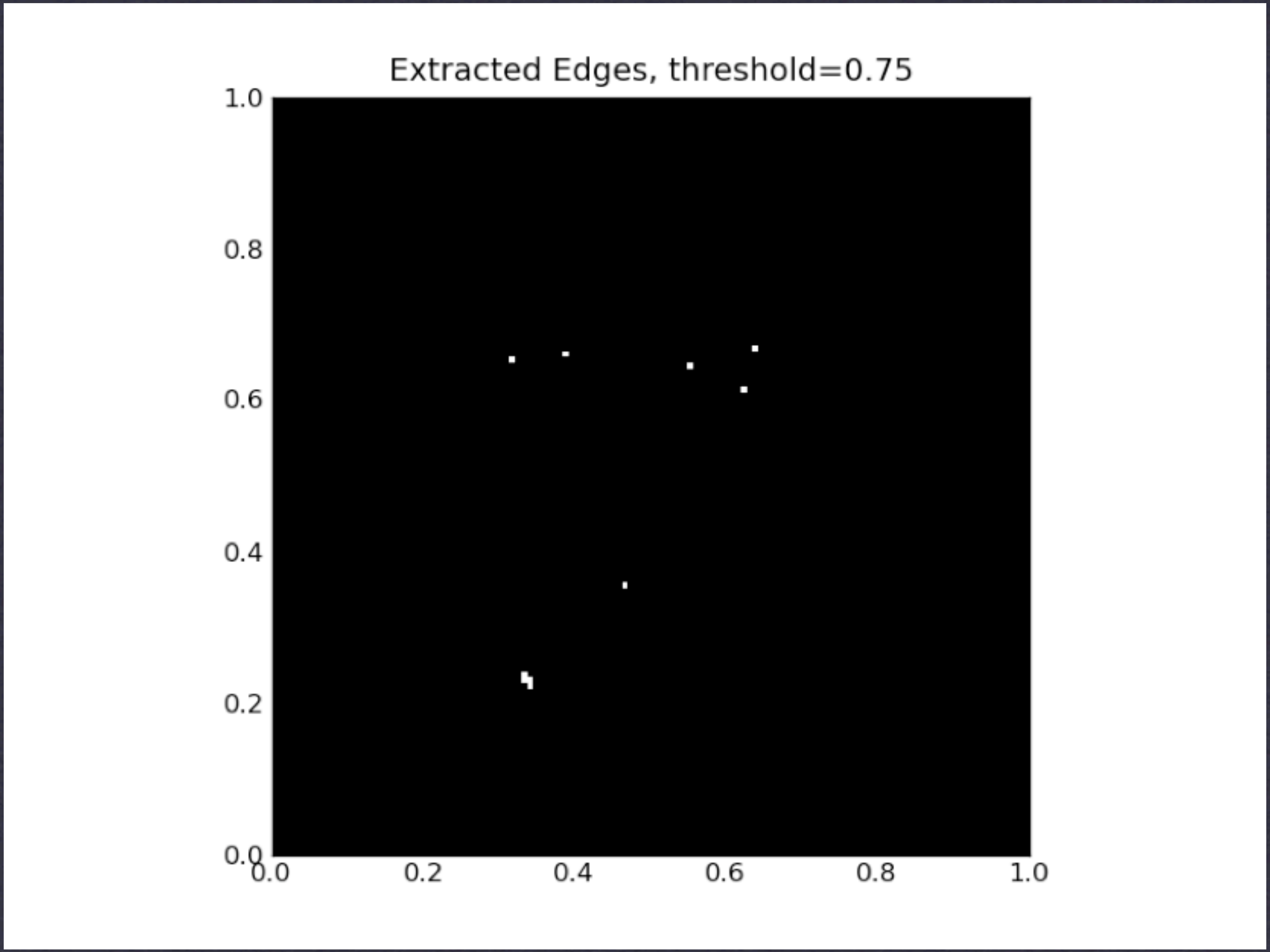

RESULT OF HIGH FREQUENCY FILTERS

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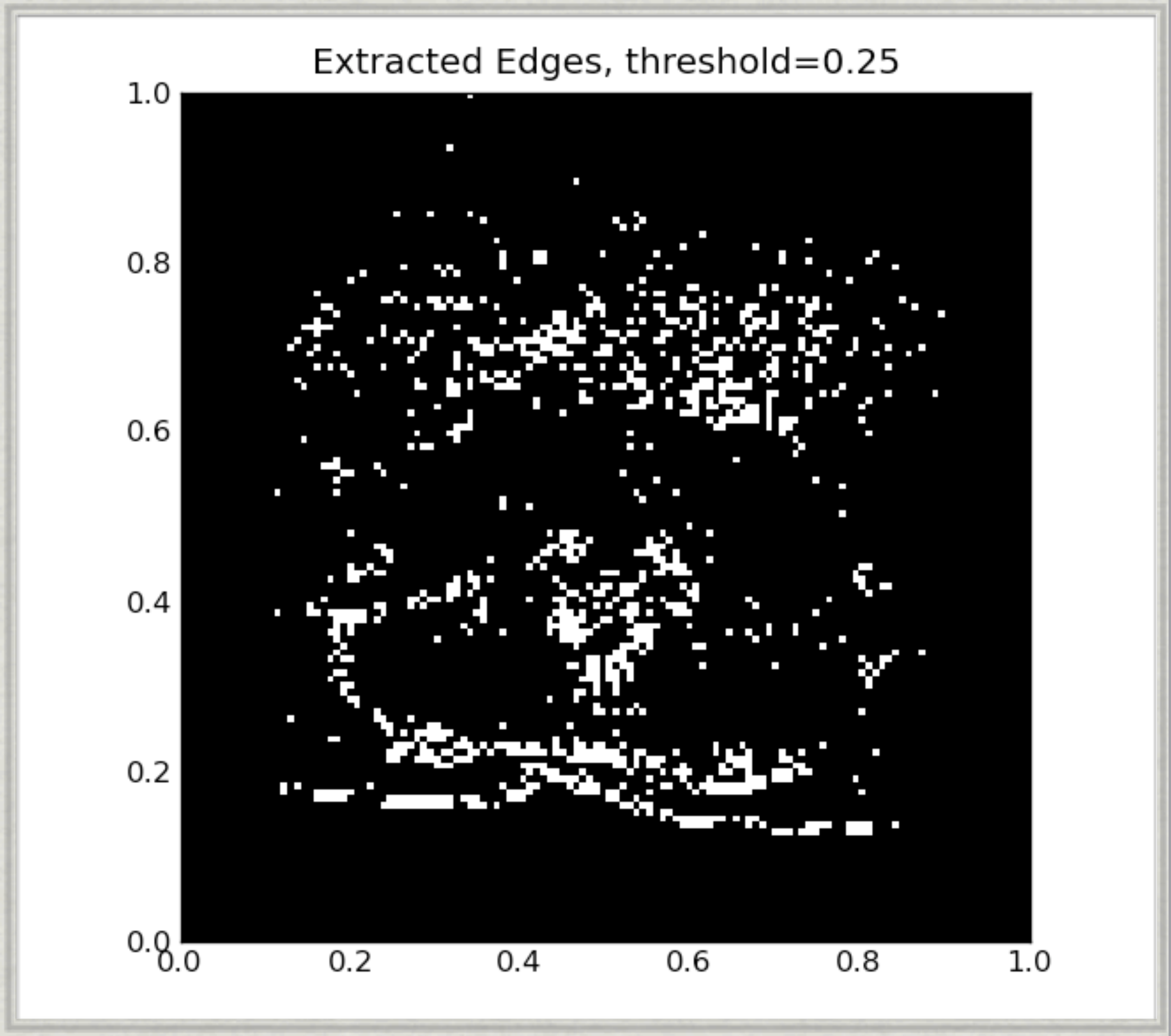
Problems

- Noisy
- Does not separate regions
- Not obvious how to "fill in the holes"



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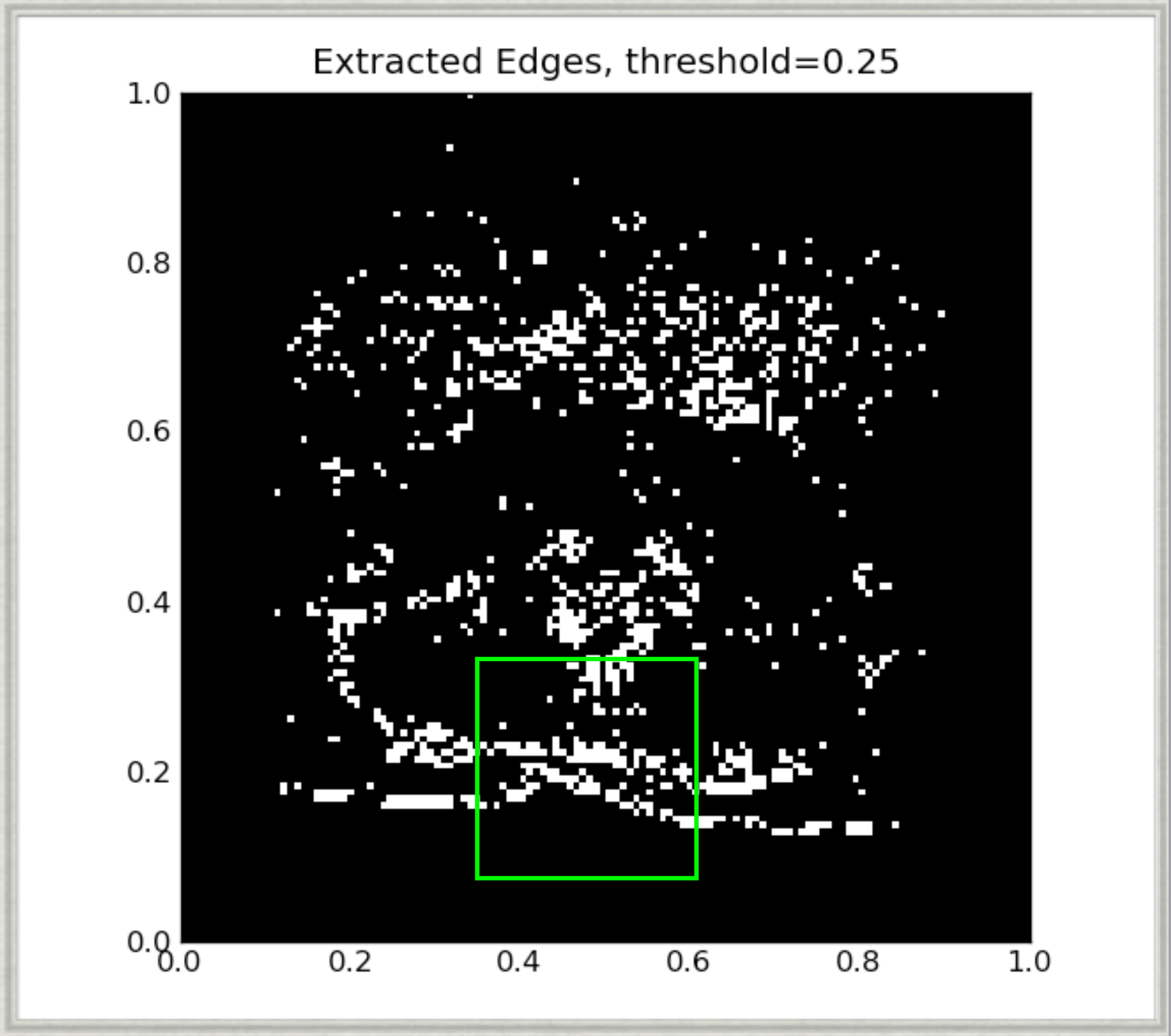


Problems

* Noisy
* Does not separate regions
* Not obvious how to "fill in the holes"

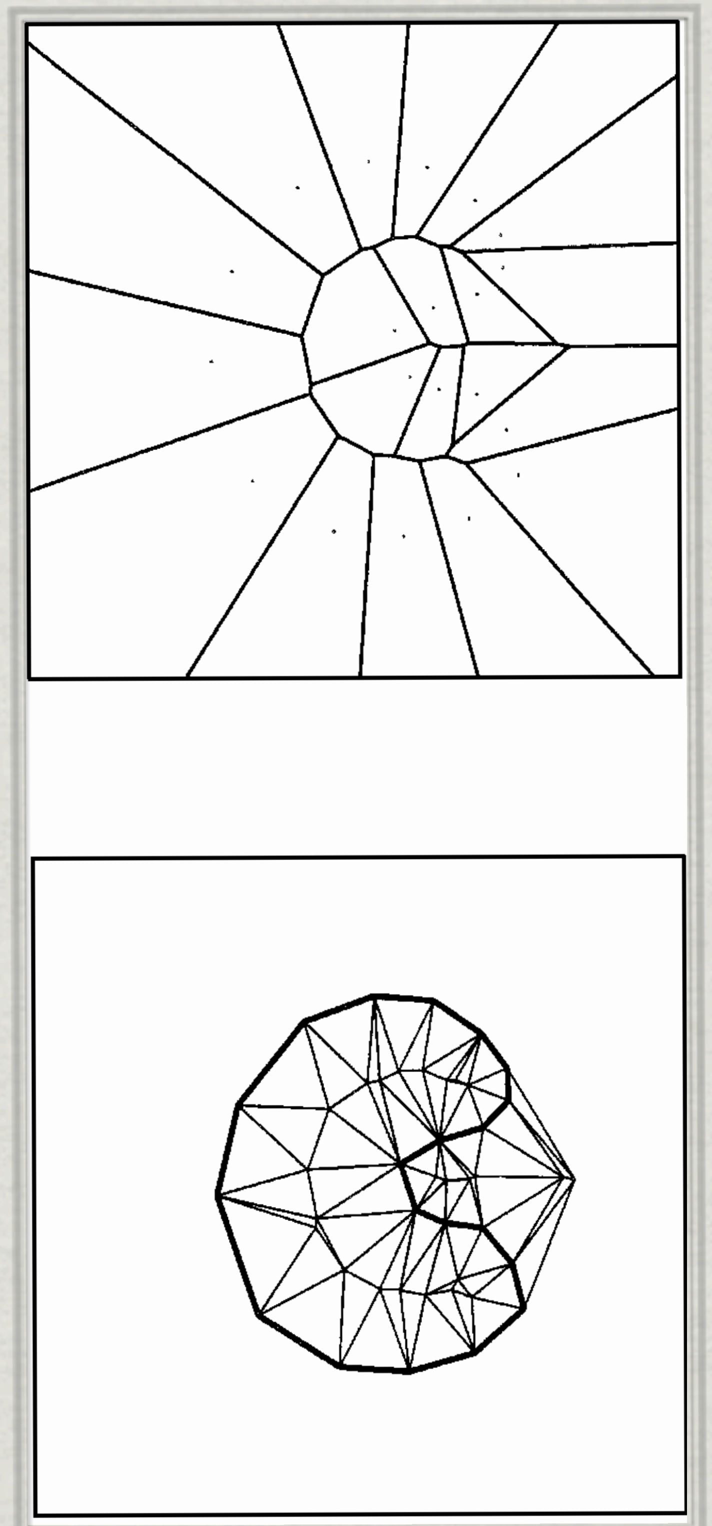
Boundary Reconstruction

- Combinatorial methods
- Active Contours/Snakes/Level Sets
- Bayesian Methods

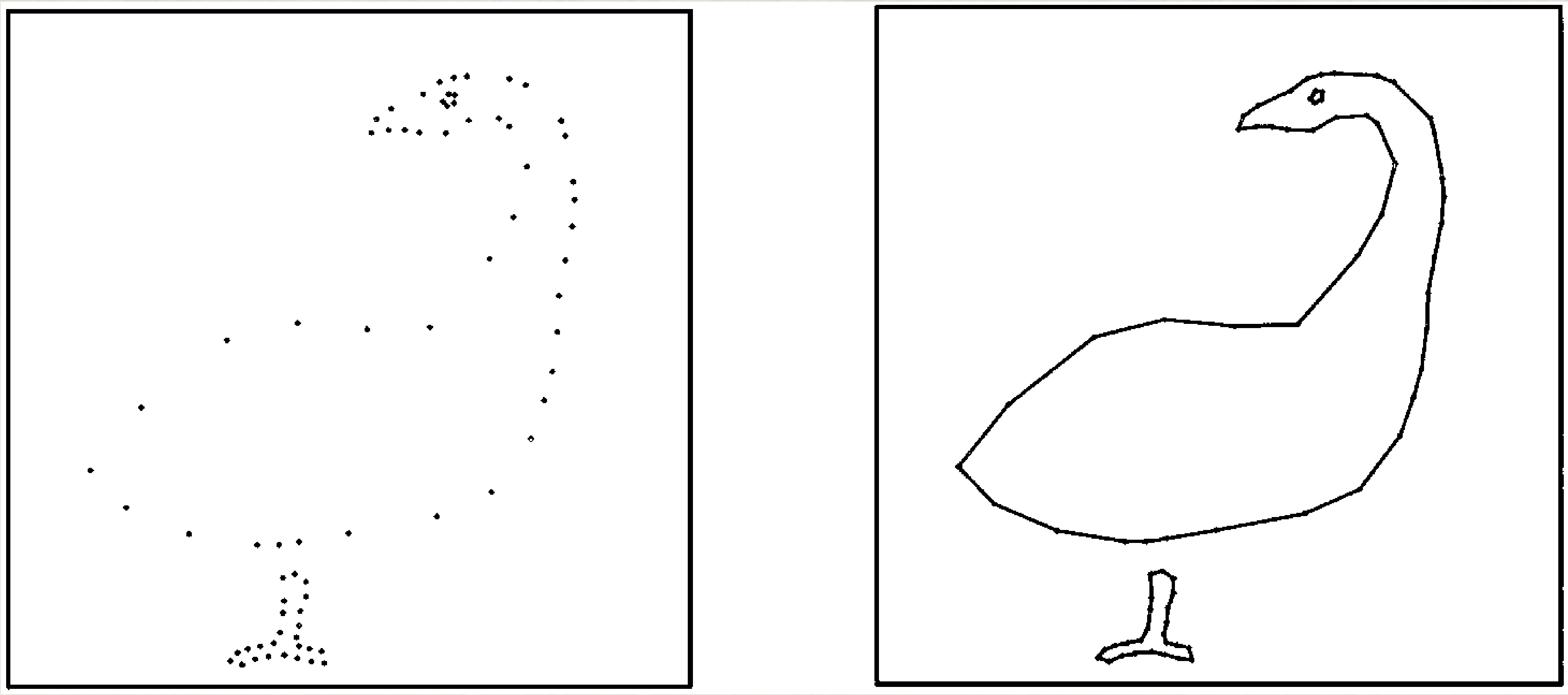
Combinatorial Methods

- Delaunay-methods: find the Crust of a point-set.
- Start with Delaunay graph.
- If a disk touches both ends of an edge in the Delaunay graph also touches a third vertex, then delete the edge.

(AMENTA, BERN, DEY, KUMAR, EPPSTEIN)

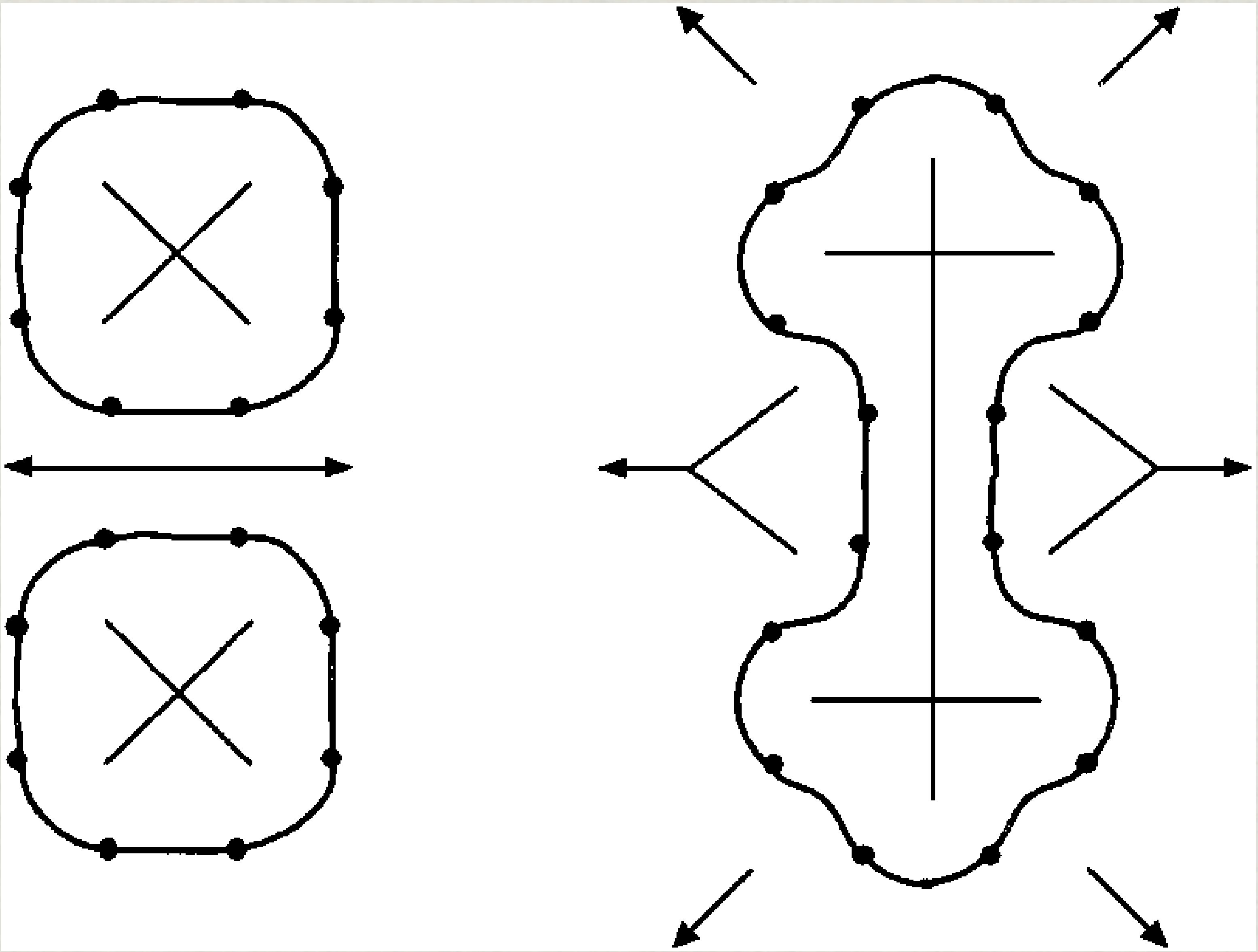


Combinatorial Methods



WIN

Combinatorial Methods



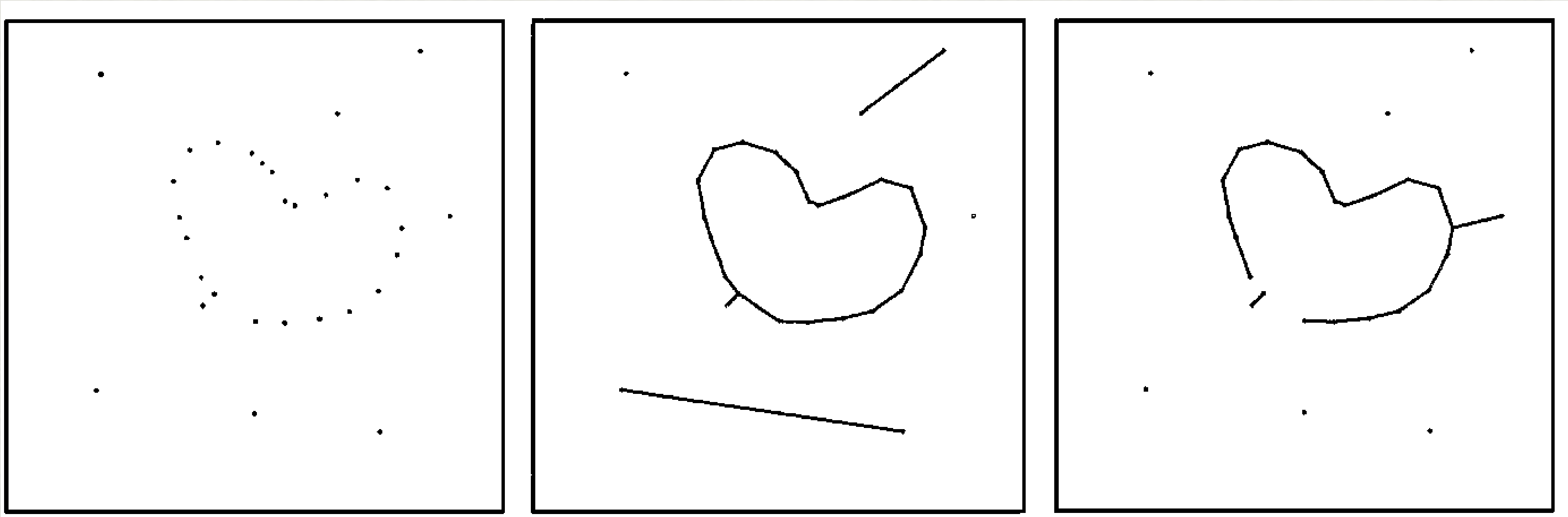
FAIL

Combinatorial Methods

- Fundamental requirements:

$$\text{sample spacing} \leq O(\text{curve separation})$$

- Sensitive to noise:



Active Contours/Snakes

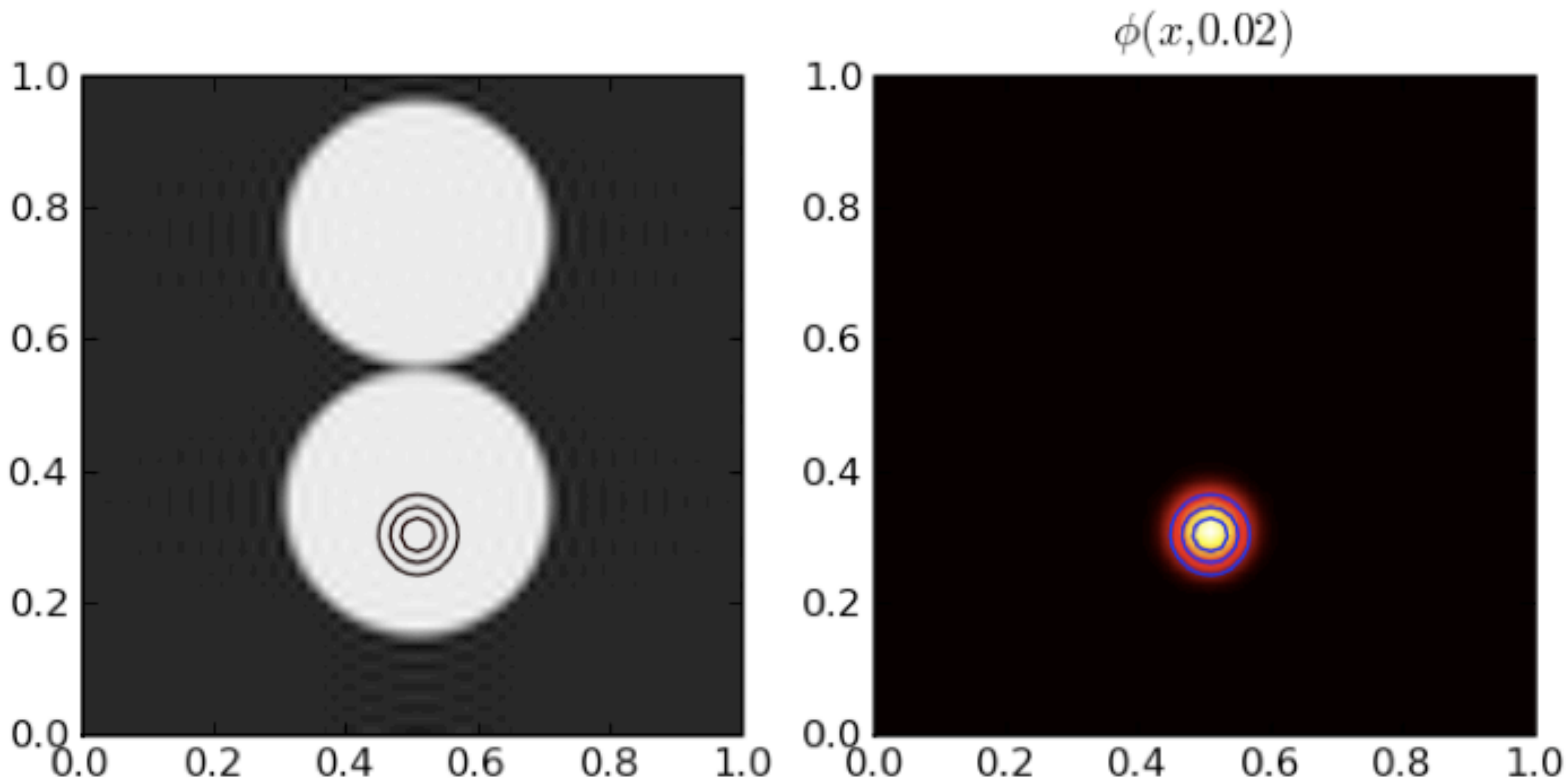
- Start with small circle
- Expand circle, stopping at edges.
- Try to maintain curve smoothness.

Level Set Segmentation

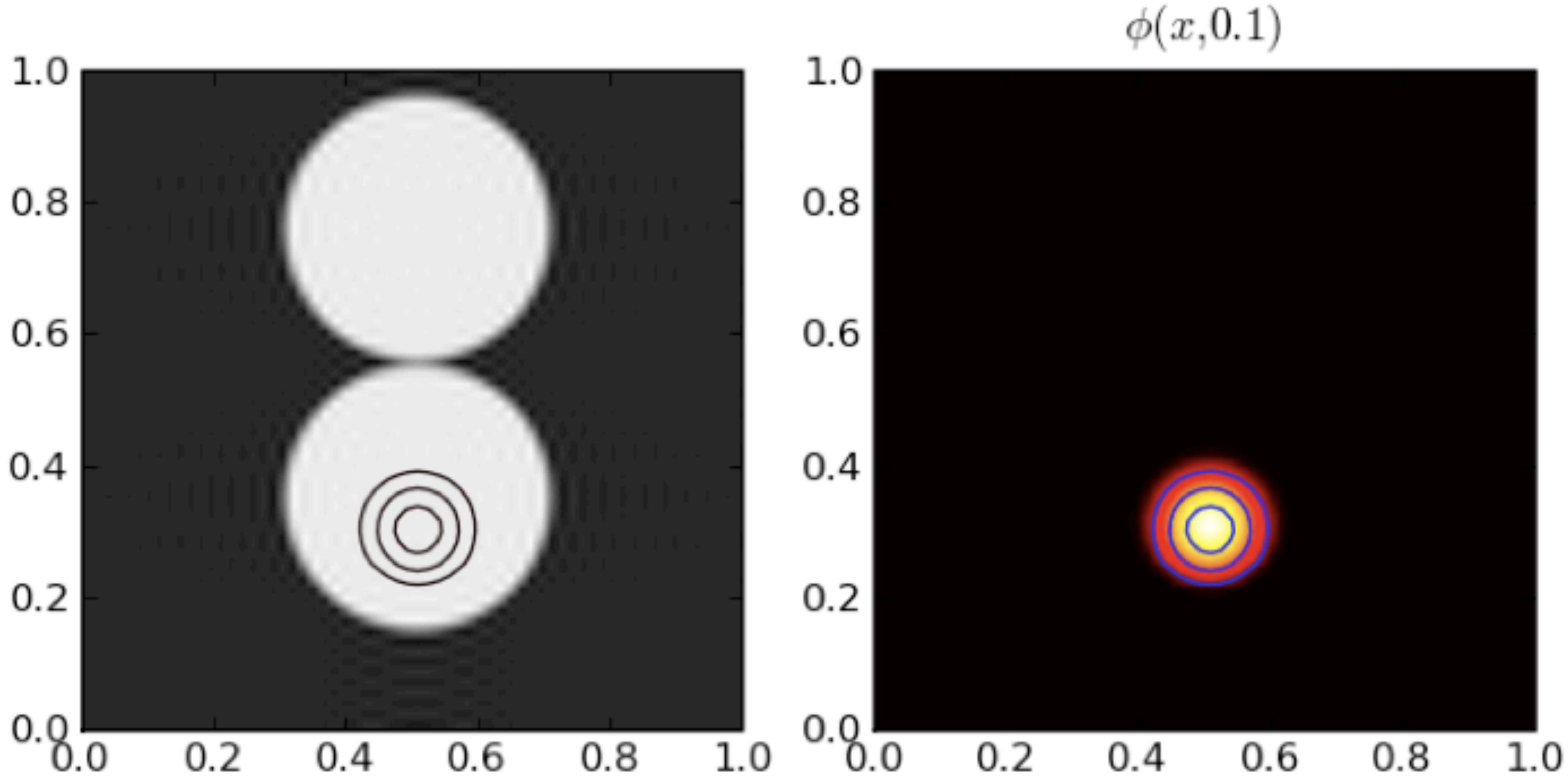
- Don't study contour directly - study level sets of auxiliary function instead.

$$\partial_t \phi(x,t) = \frac{|\nabla \phi(x,t)|}{1+\alpha E(x)} f(\phi(x,t)-1)/2 + 2\Delta\phi(x,t) + \text{regularization}$$

- E(x) is result of edge detectors.



LEVEL SET SEGMENTATION



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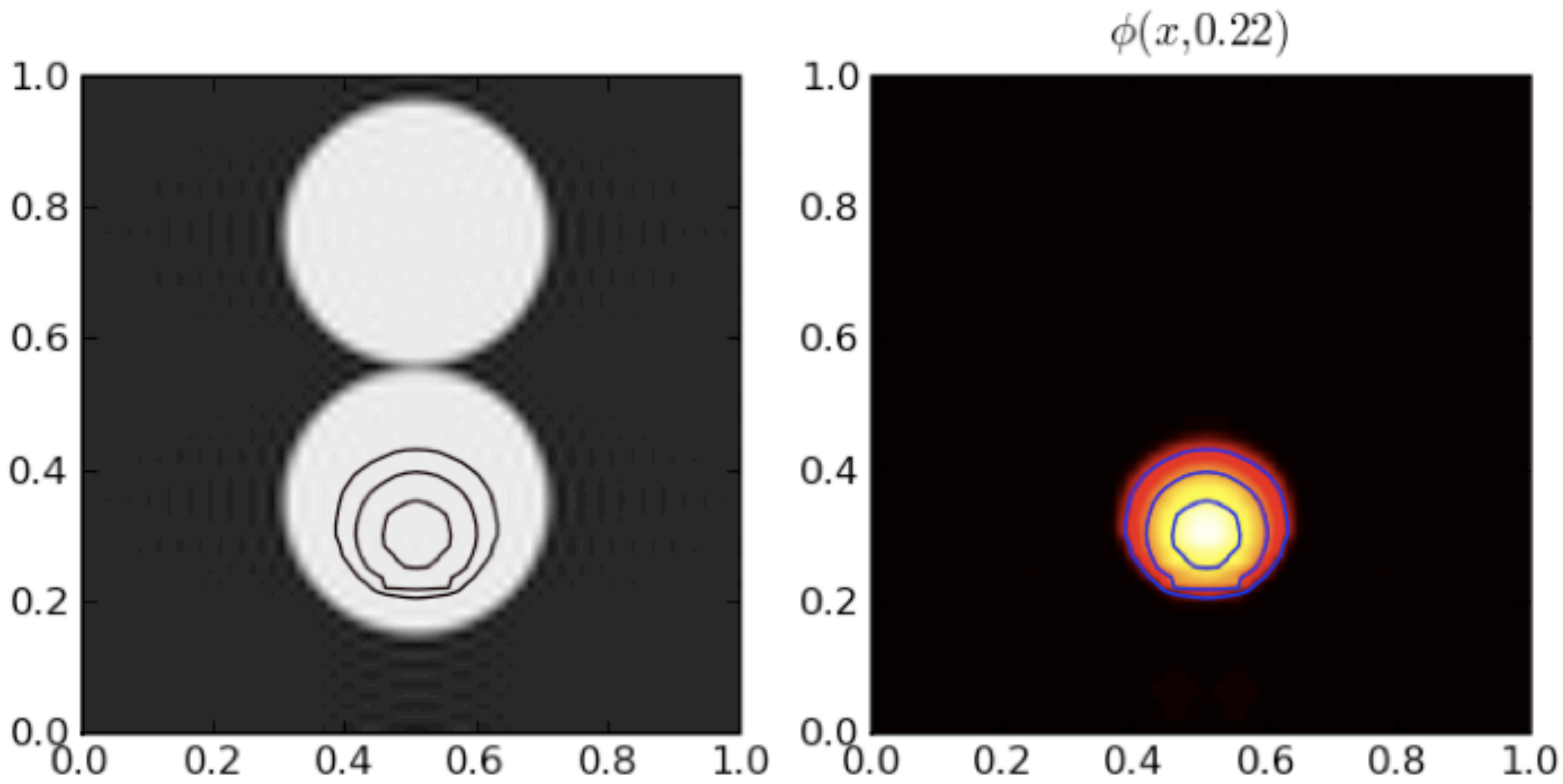

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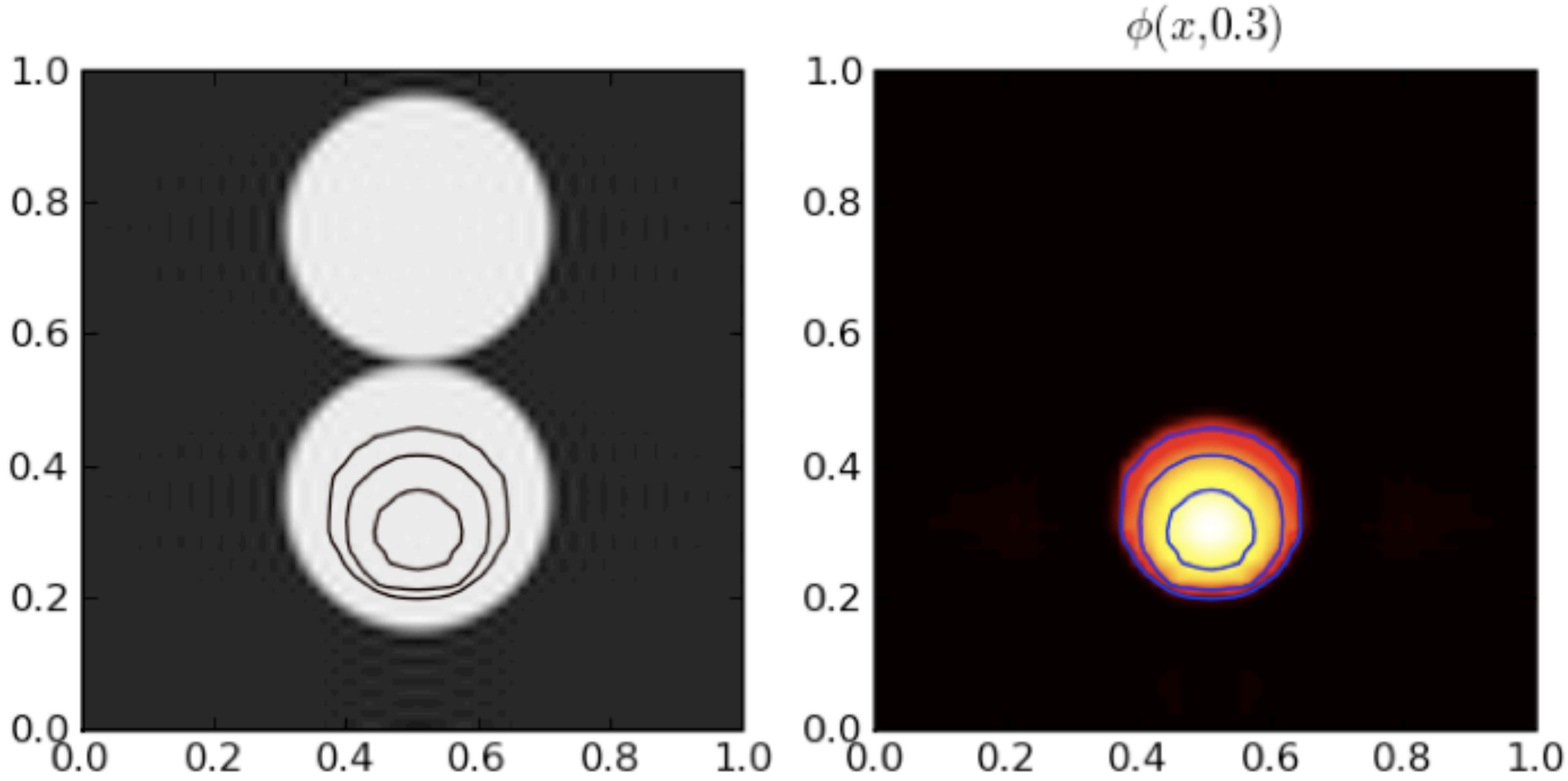

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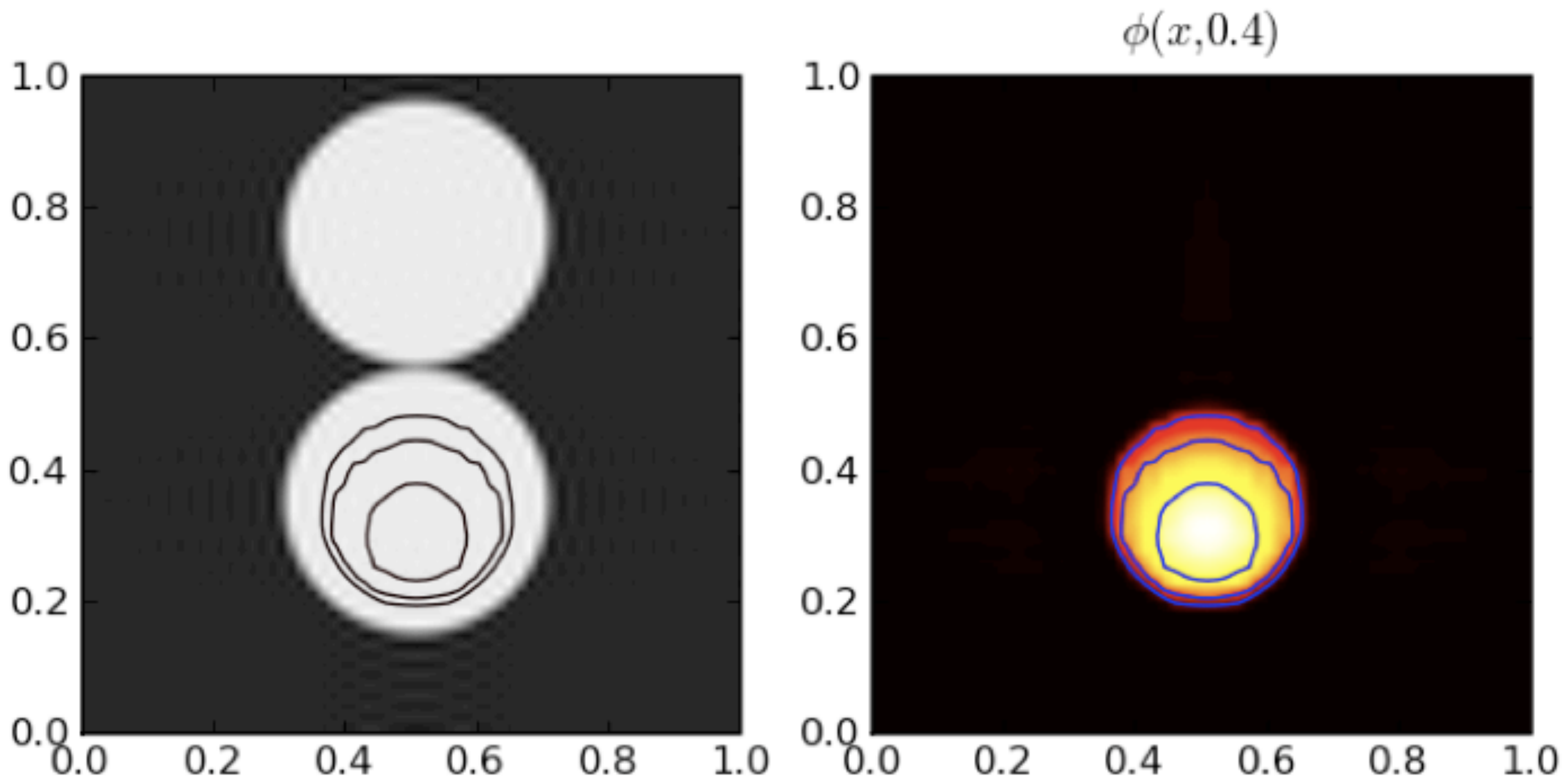

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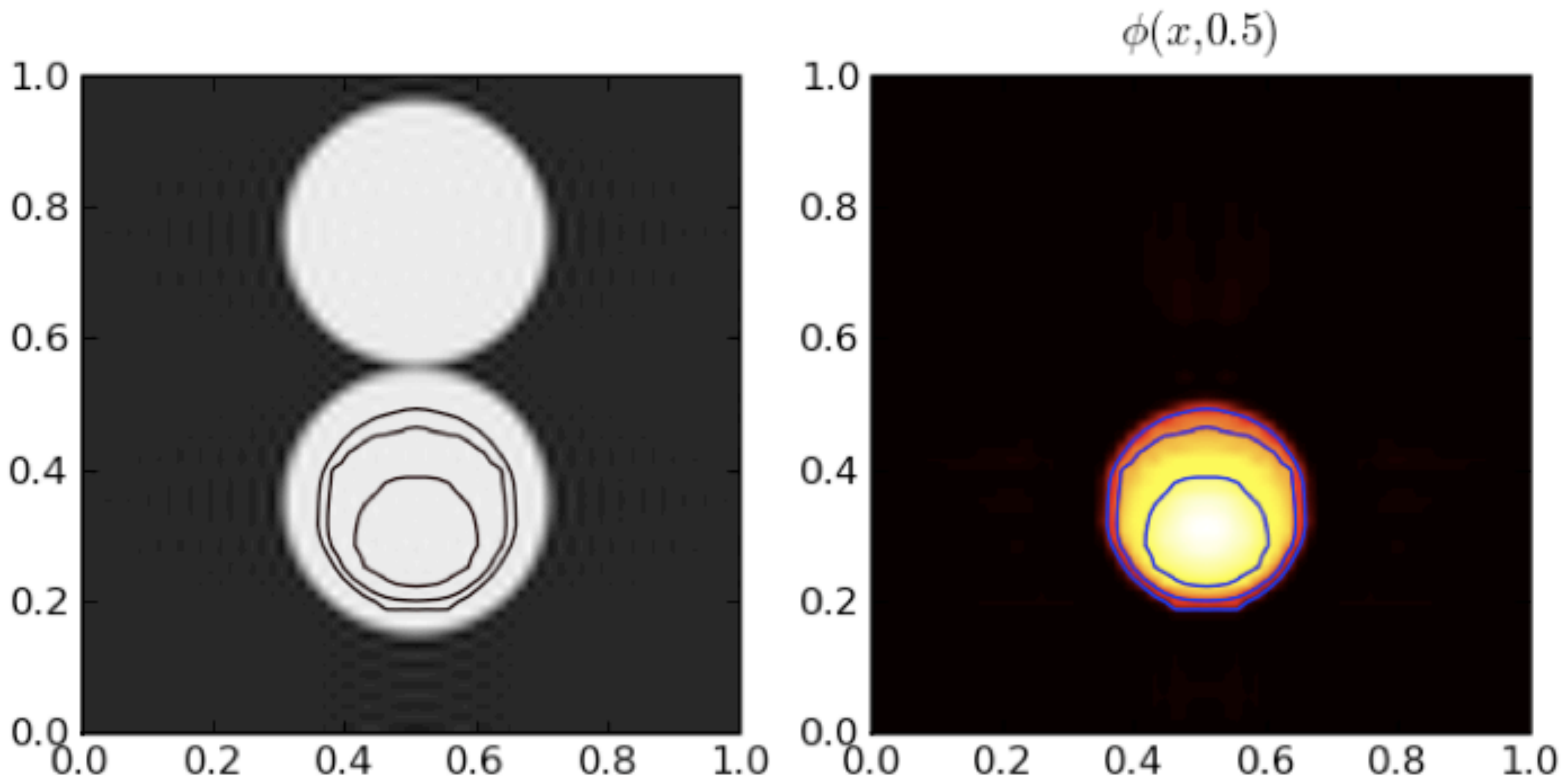

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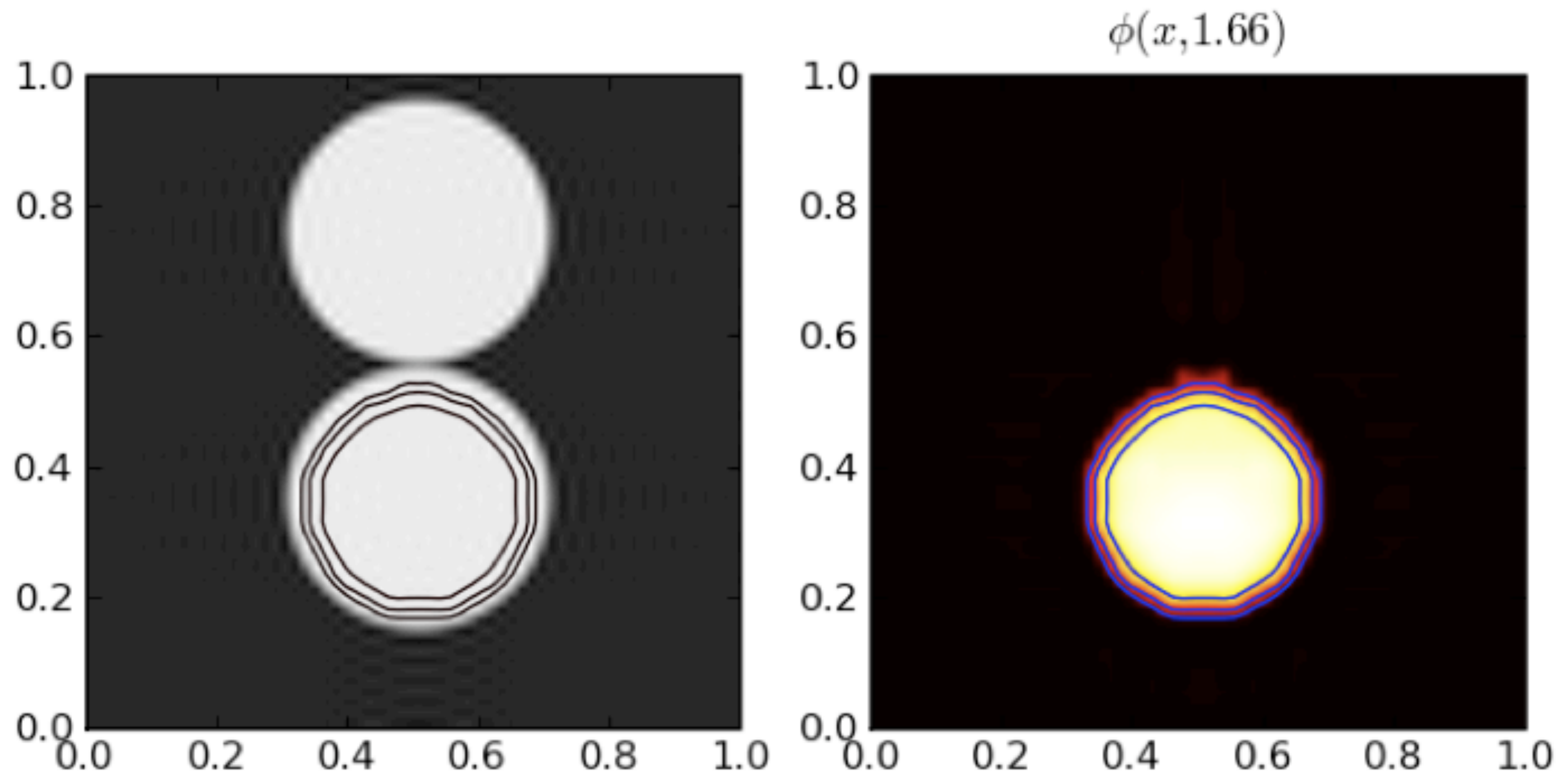

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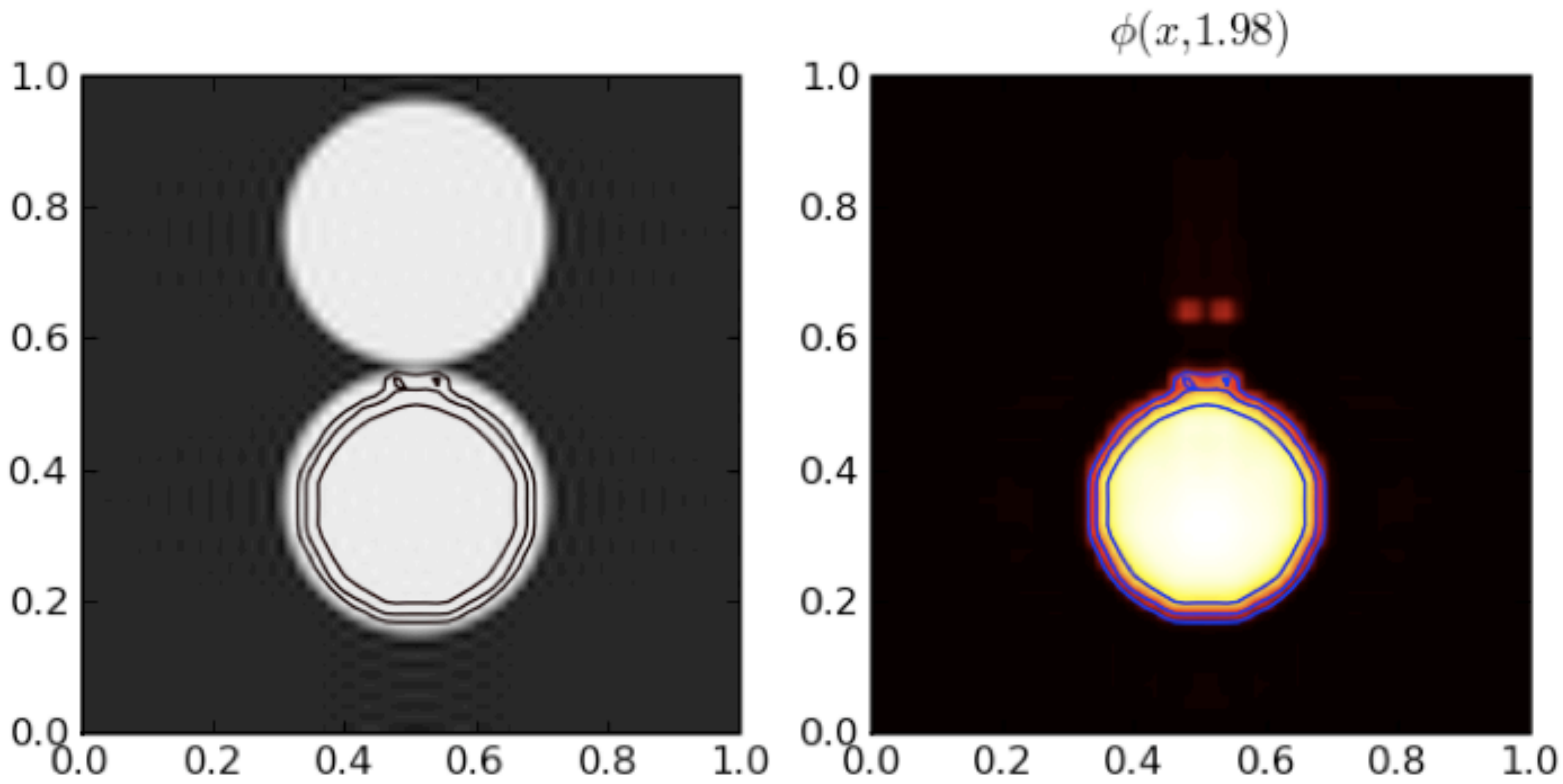

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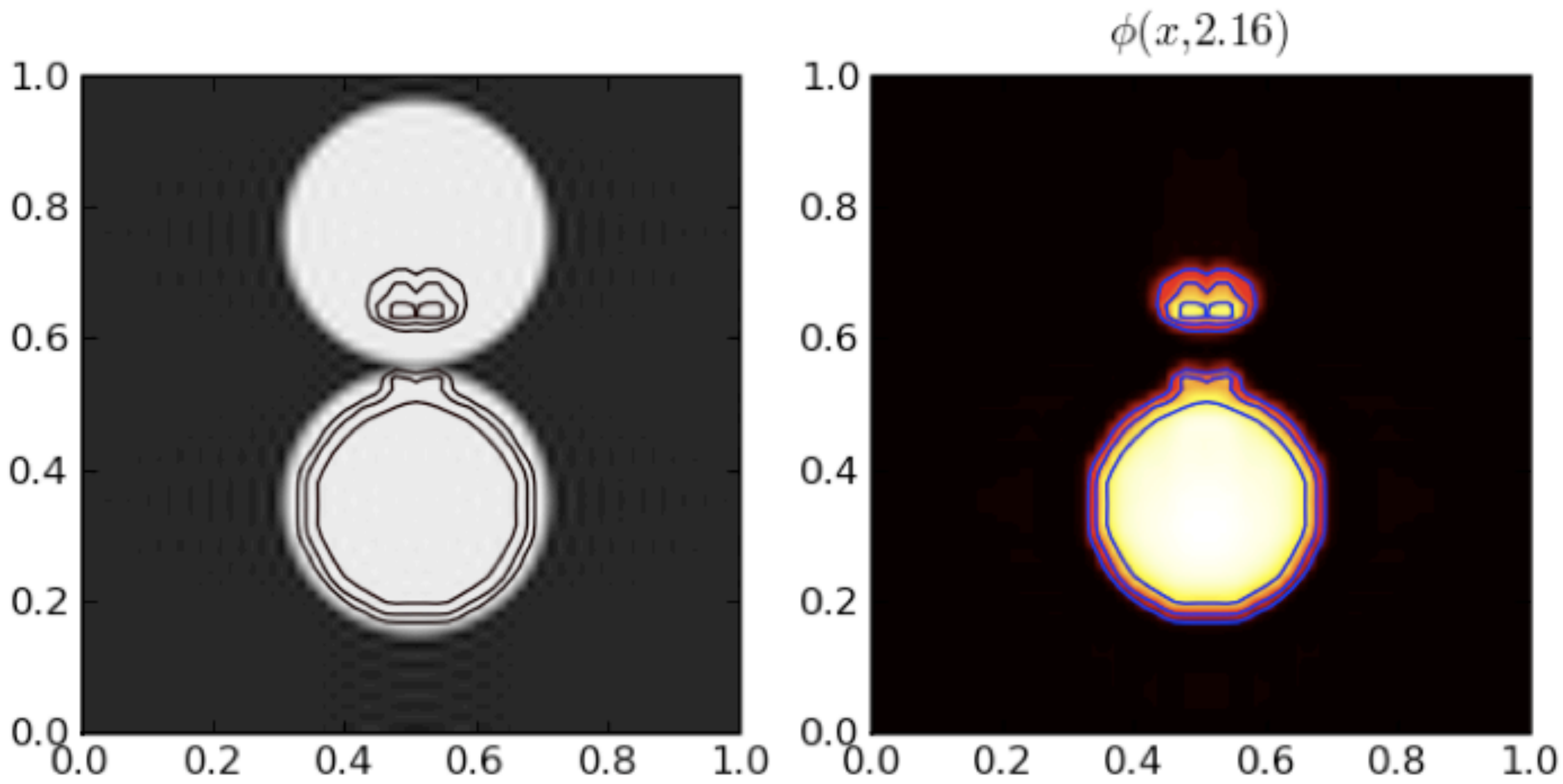

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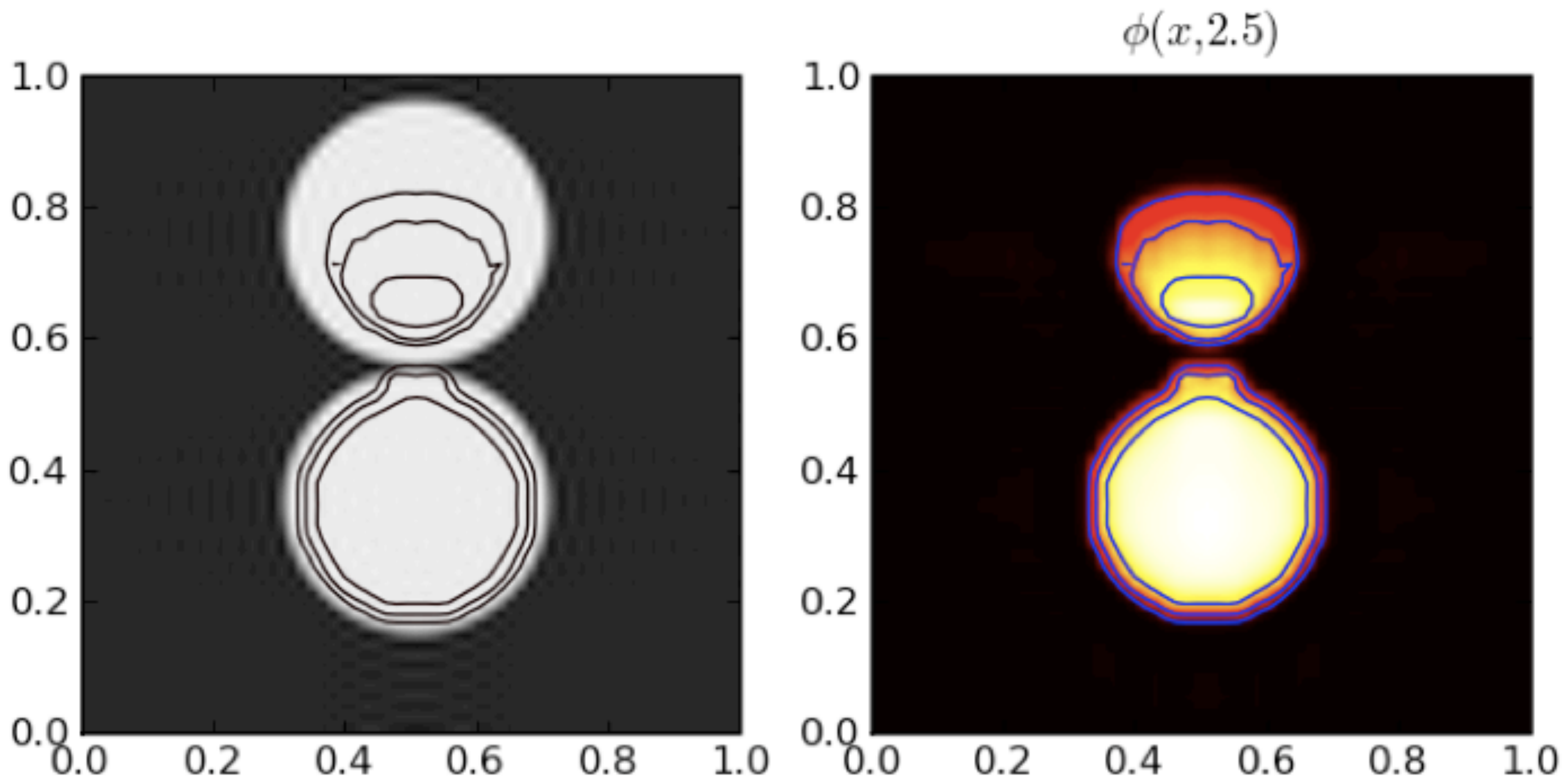


LEVEL SET SEGMENTATION

2 dimensions is not 1 dimension “done twice”

Parameterize images

$$\boldsymbol{\rho}(x) = \left[\sum_{j=0}^{M-1} \boldsymbol{\rho}_j 1_{\gamma_j}(x)\right] + \boldsymbol{\rho}_{\mathrm{tex}}(x)$$

Parametric Model

- Segmentation problem:

$$\text{Find: } M, \gamma_j(t)$$

- Reconstruction problem:

$$\text{Find: } M, \gamma_j(t), \rho_j, \rho_{\text{tex}}(x)$$

Singular Support

- Edges are the *singular support* of the function:

$$\forall \lambda > 0, \sup_{|\vec{k}| \geq k_r} \left| \int e^{i\vec{k}\cdot x} \boldsymbol{\rho}(x) \chi((x - x_0)\lambda) dx \right| = O({k_r}^{-3/2})$$

- Singular support is set of points

$$\vec{x}_0 = \gamma_j(t)$$

Wavefront Set

- Singular support extends to *wavefront* in higher dimensions

$$\forall \lambda > 0, \sup_{r \geq k_r} \left| \int e^{irk_0 \cdot x} \boldsymbol{\rho}(x) \chi((x - x_0)\lambda) dx \right| = O({k_r}^{-3/2})$$

- Wavefront is set of *surfels*

$$(\vec{x}_0, \vec{k}_0) = (\gamma_j(t), \pm N_j(t))$$

2D is not 1D squared

$$\text{singular support} \subset \mathbb{R}^N$$

$$\text{wavefront} \subset \mathbb{R}^N \times (\mathbb{S}^{N-1}/\{\pm 1\})$$

$$\mathrm{P}_x \text{wavefront} = \text{singular support}$$

2D is not 1D squared

$$\mathbb{R}^1 \times \mathbb{R}^1 \neq \mathbb{R}^2 \times (\mathbb{S}^1/\{\pm 1\})$$

Wavefront Detection

Wavefront Detectors

- What does the Fourier transform of an edge look like?

Wavefront Detectors

- Calculate with Green's Theorem

$$\widehat{1_{\gamma_j}}(\vec{k}) = \int\int_{\Omega_j} e^{i\vec{k}\cdot x} dx_1 dx_2 = \int\int_{\Omega_j} \partial_{x_1} F_2(\vec{k},x) - \partial_{x_2} F_1(\vec{k},x) dx_1 dx_2$$

$$= \int_{\mathbb{S}^1} F(\vec{k},\gamma_j(t)) \cdot \frac{d\gamma_j(t)}{dt} dt = \frac{1}{i|\vec{k}|^2} \int_{\mathbb{S}^1} e^{i\vec{k}\cdot\gamma_j(t)} \vec{k}^{\perp} \cdot \gamma_j'(t) dt$$

(HAT TIP: EUGENE SORETS)

Wavefront Detectors

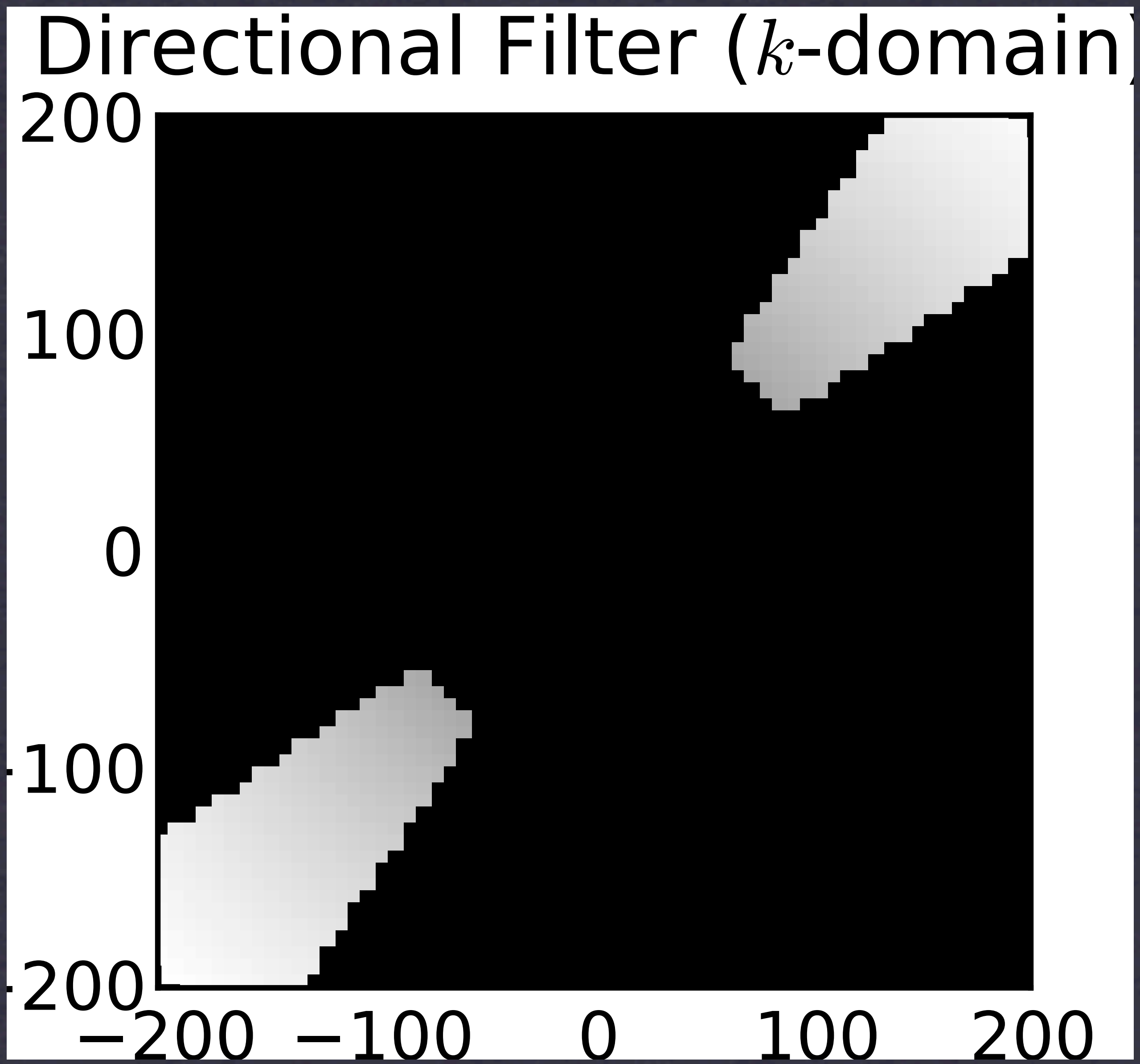
- Phase stationary when $\vec{k} \cdot \gamma'(t) = 0$

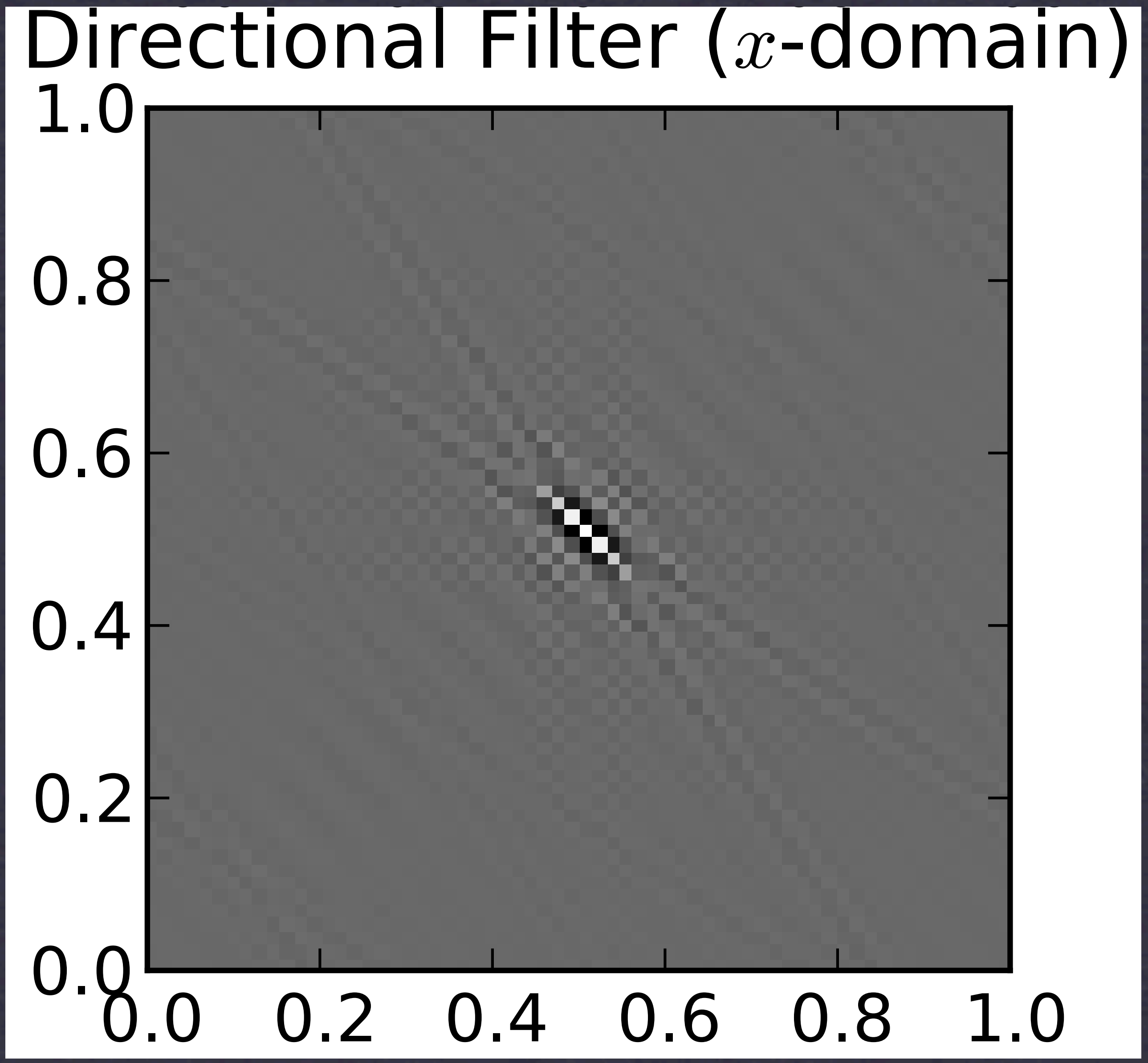
$$\sum_{j=0}^{M-1} \boldsymbol{\rho}_j \widehat{1_{\gamma_j}}(\vec{k}) = \sum_{j=0}^{M-1} \boldsymbol{\rho}_j \left[\frac{e^{i\vec{k}\cdot\gamma(t_j(\vec{k}))}}{\left|\vec{k}\right|^{3/2}} \frac{\sqrt{\pi}}{\sqrt{\kappa_j(t_j(\vec{k}))}} + \frac{e^{i\vec{k}\cdot\gamma(t_j(-k))}}{\left|\vec{k}\right|^{3/2}} \frac{\sqrt{\pi}}{\sqrt{\kappa_j(t_j(-\vec{k}))}} \right] + O({k_r}^{5/2}) \quad (2.2)$$

$$t_j(\vec{k}) \text{ satisfies } \vec{k} \cdot \gamma'(t_j(\vec{k})) = 0$$

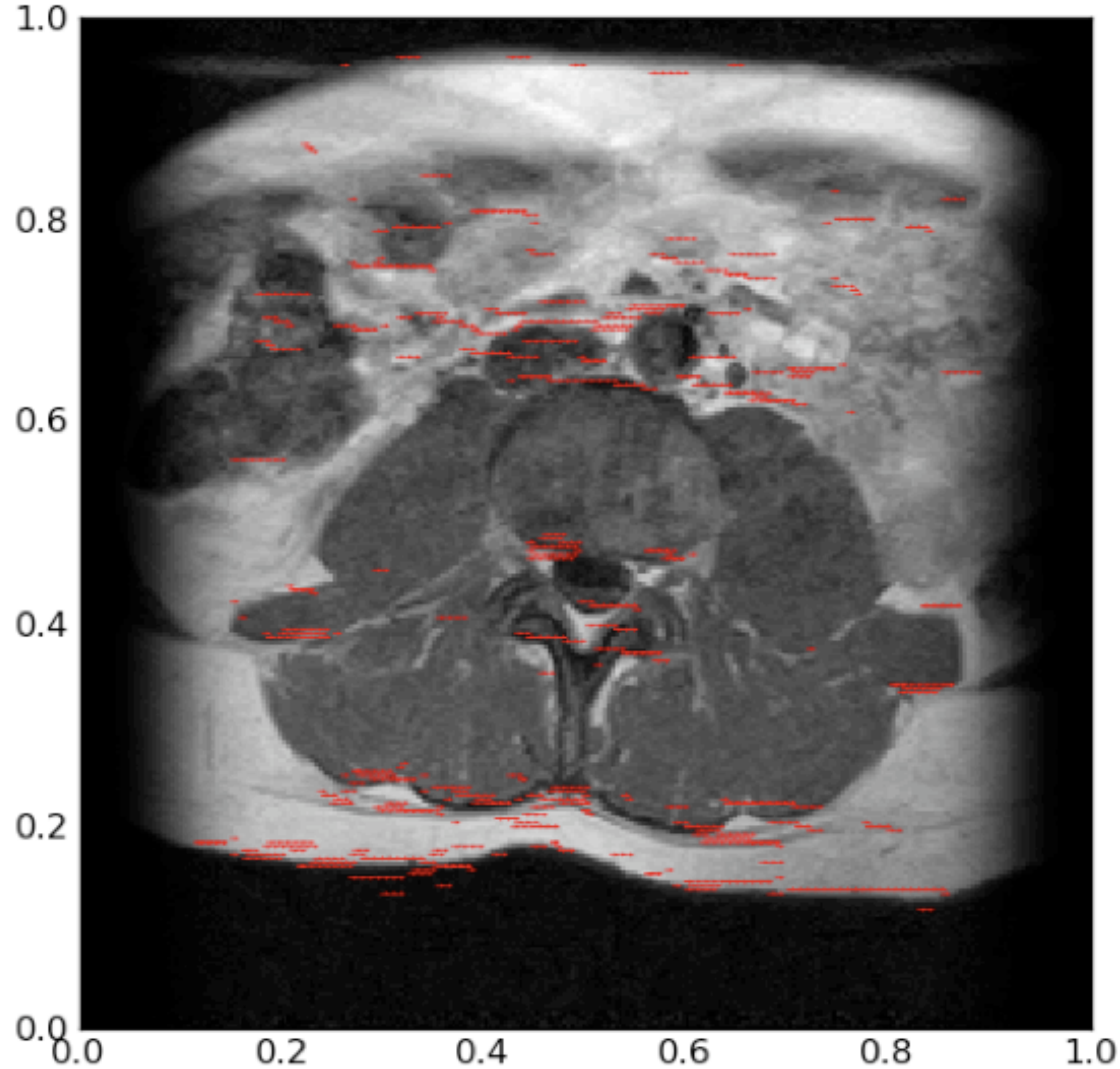
Wavefront Detectors

- Ray $\vec{k} = k_r \vec{k}_\theta$ encodes location of edges with normals pointing in direction $\vec{k}_\theta$
- Localizing on this region yields surfels in the wavefront pointing in direction $\vec{k}_\theta$



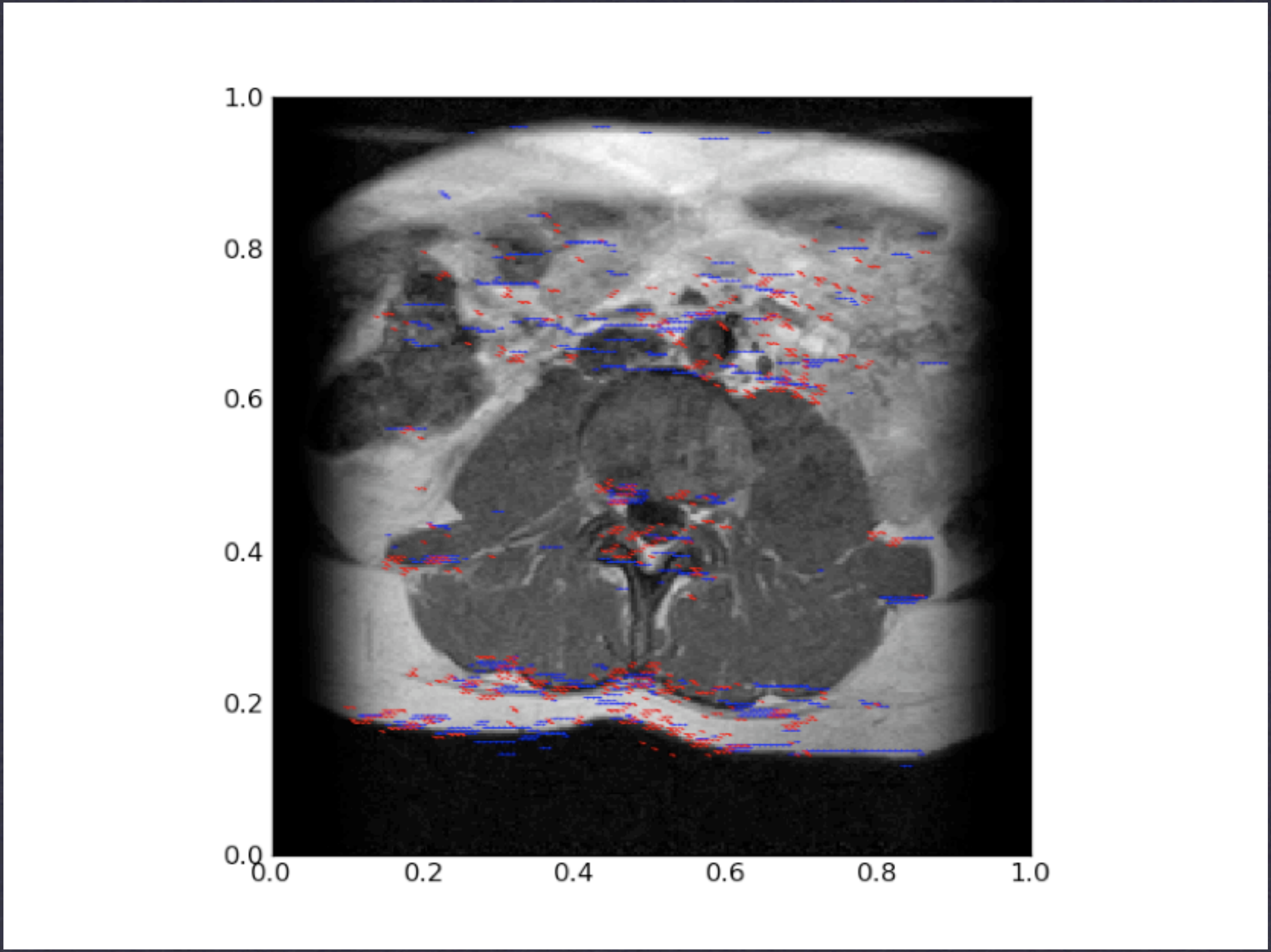


DIRECTIONAL FILTERS



WAVEFRONT FILTERS

ARROWS ARE TANGENTIAL TO THE EDGE



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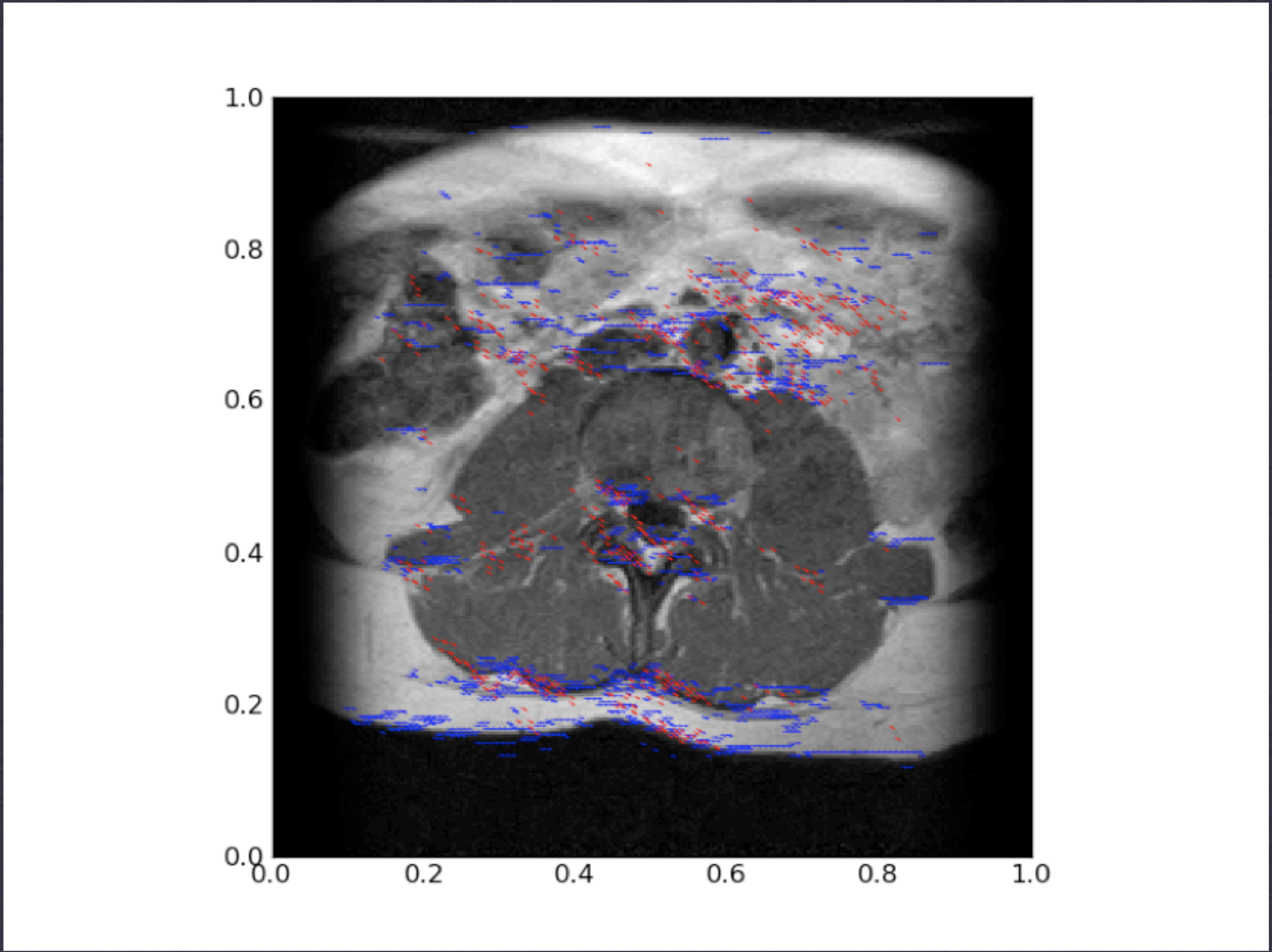

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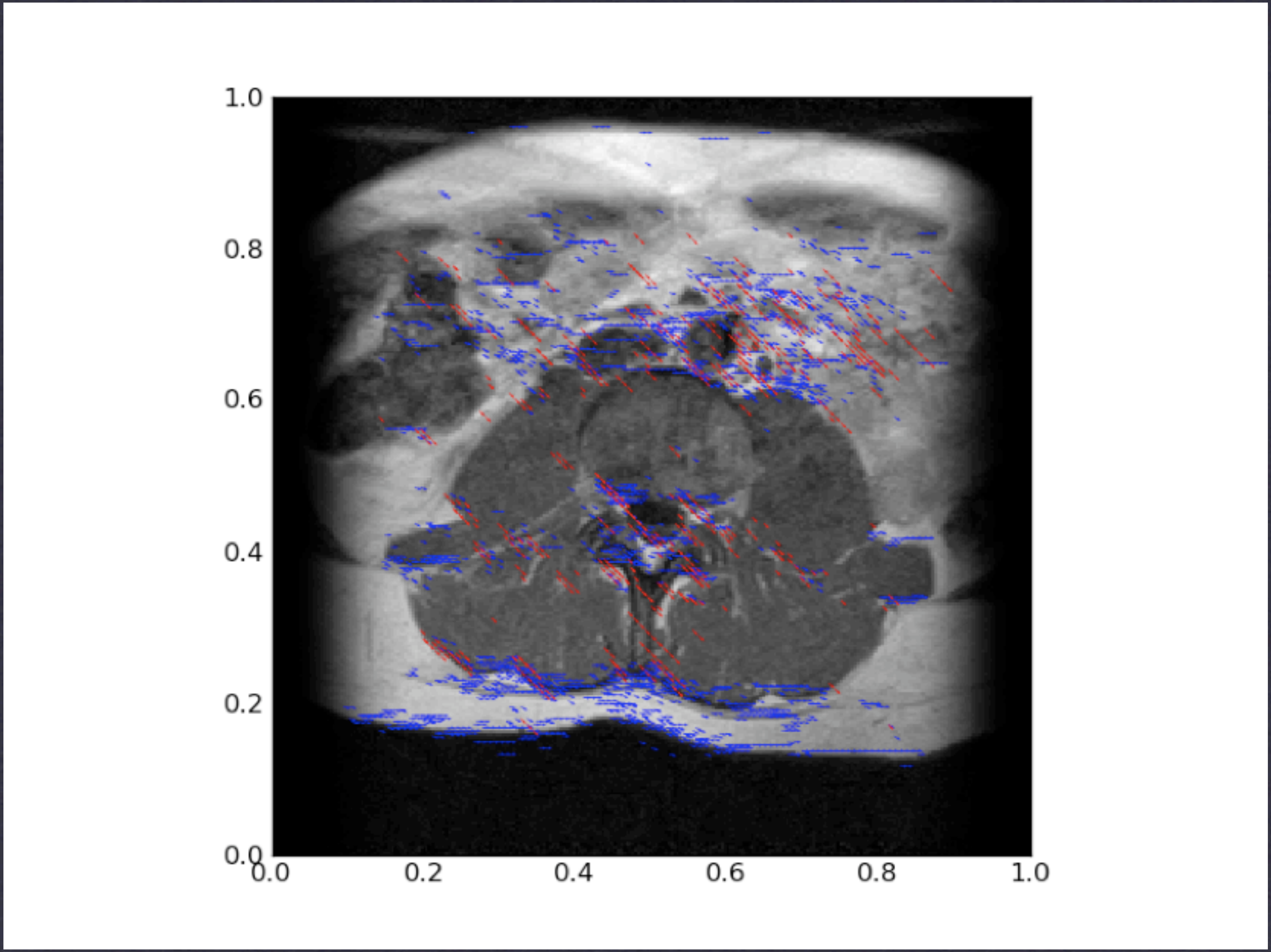

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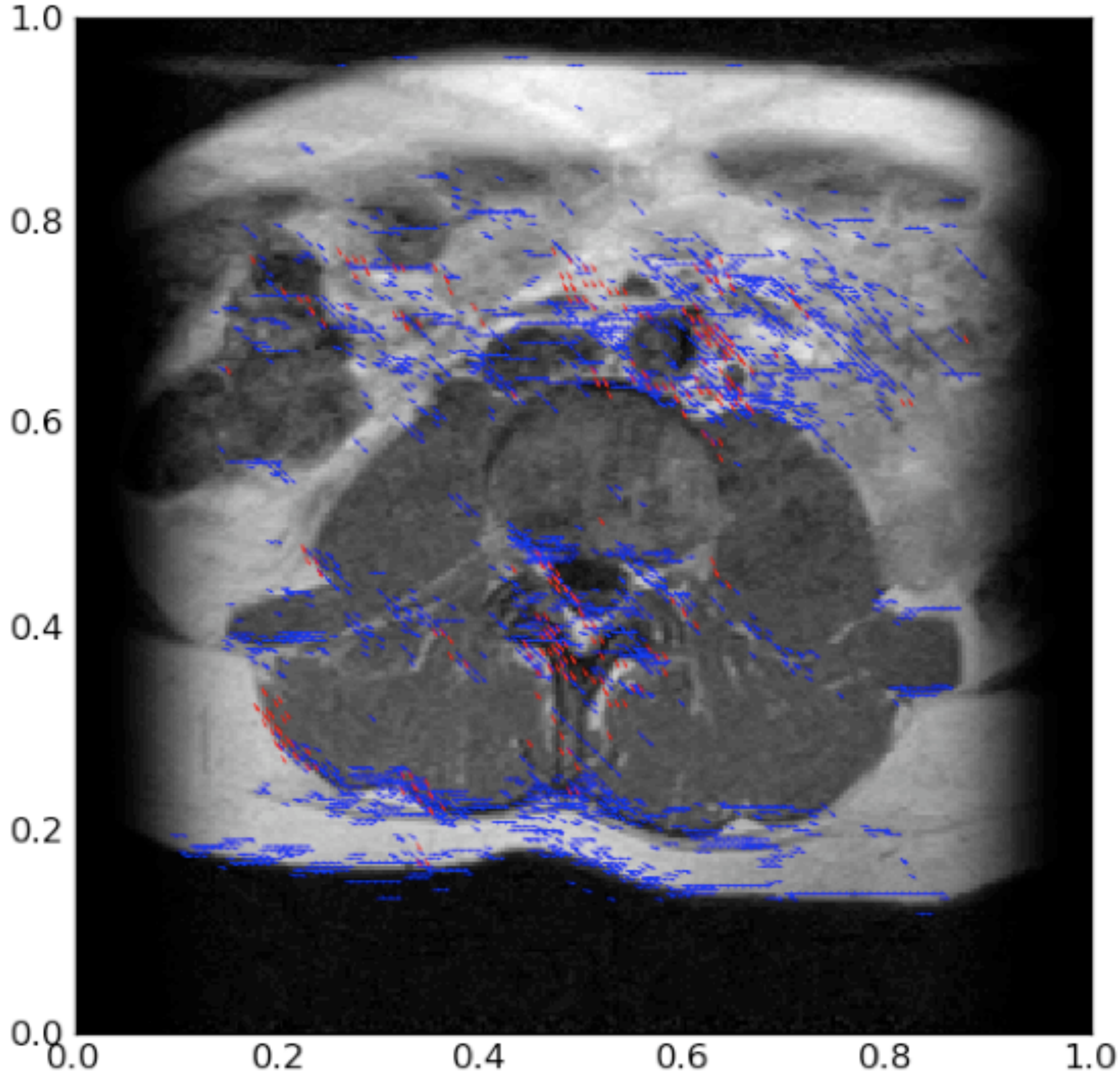

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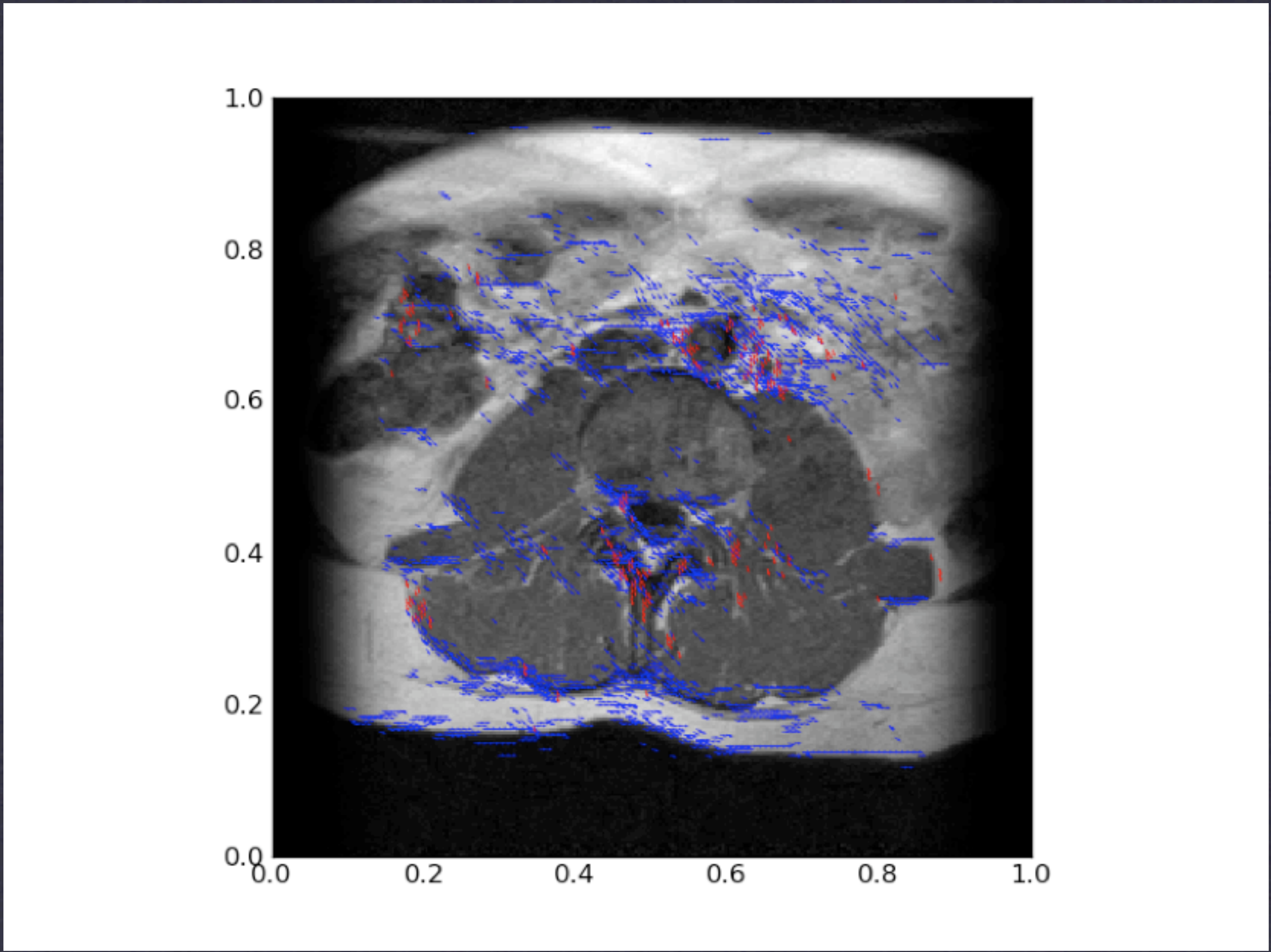

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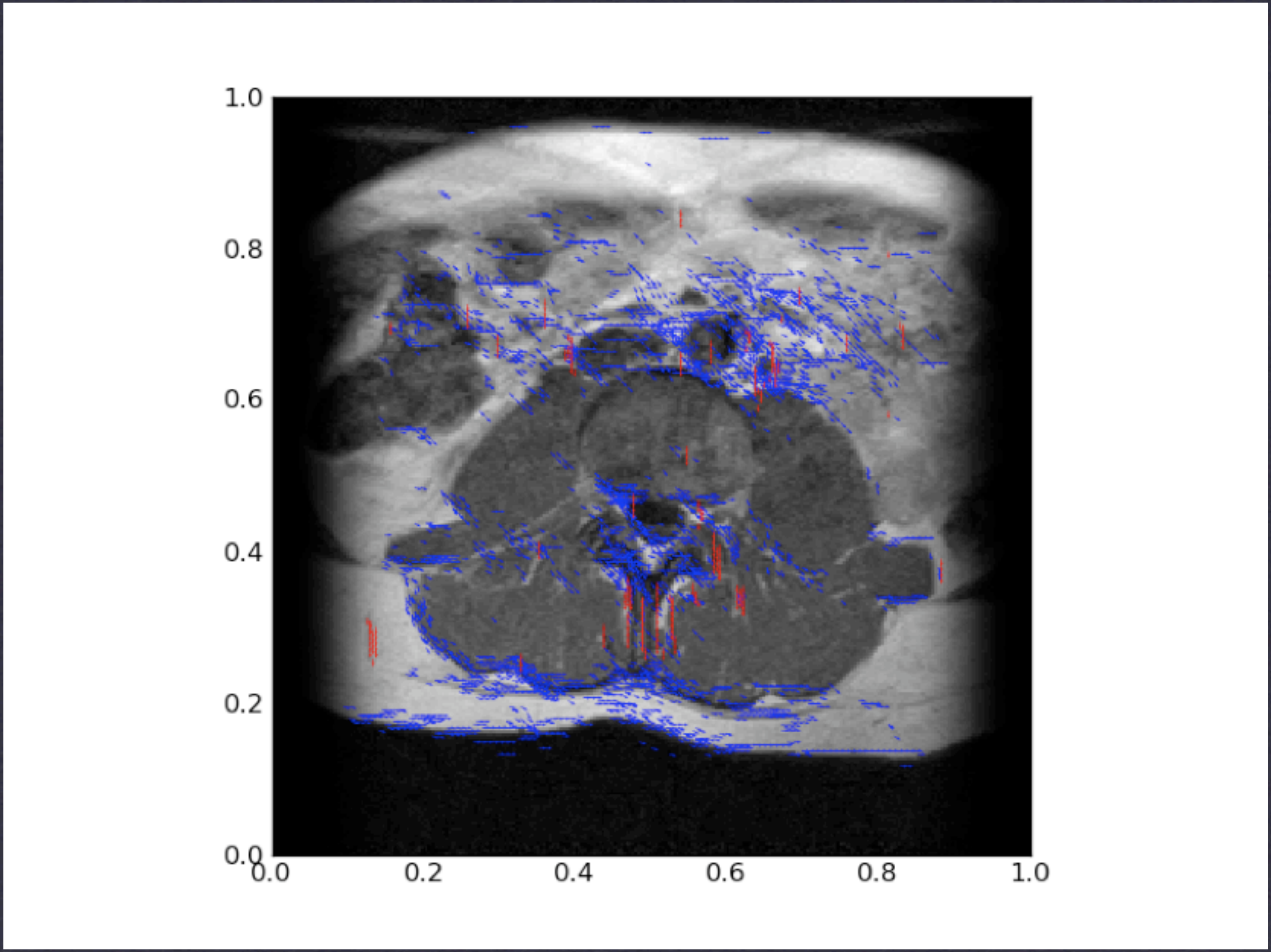

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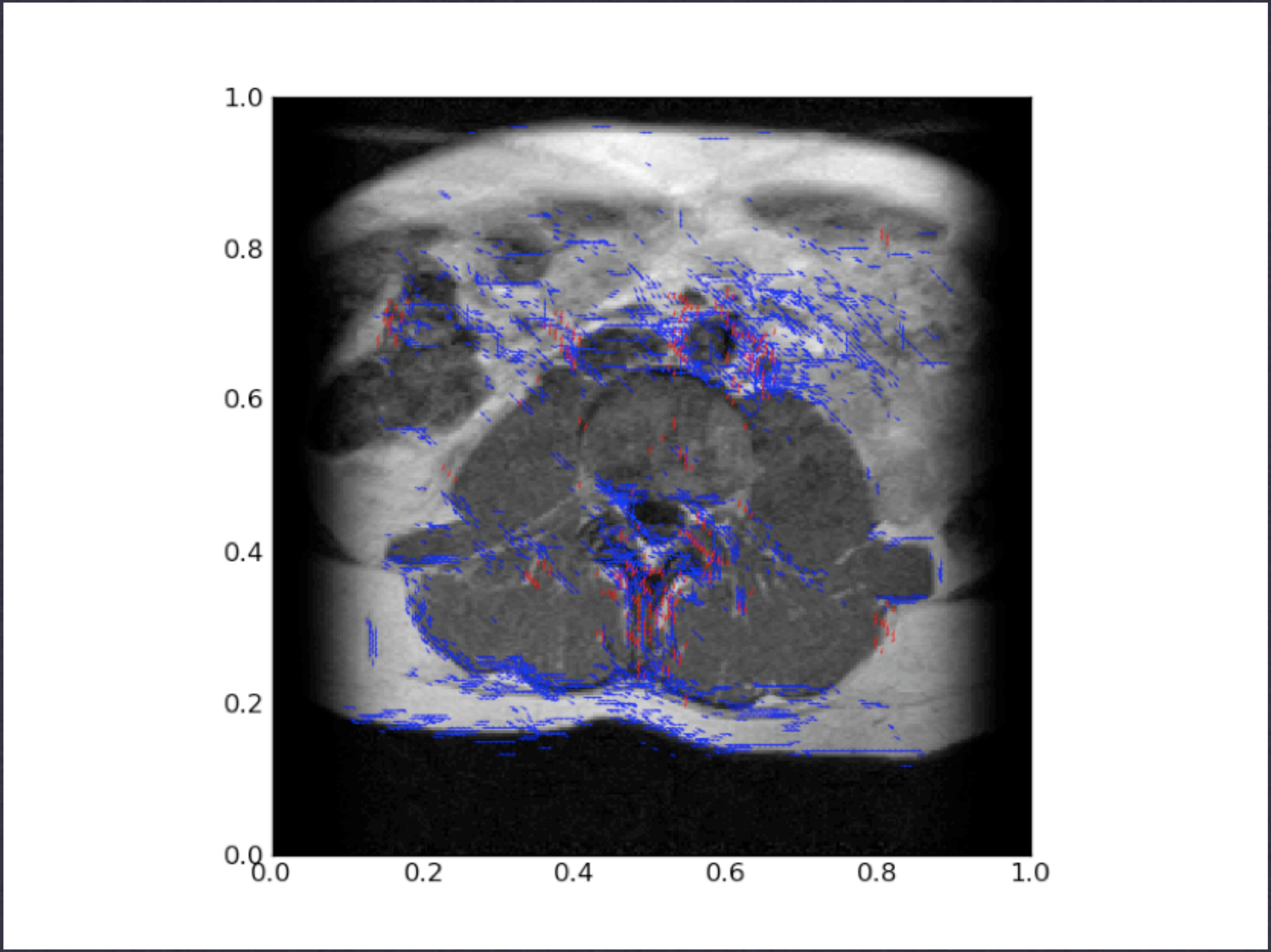
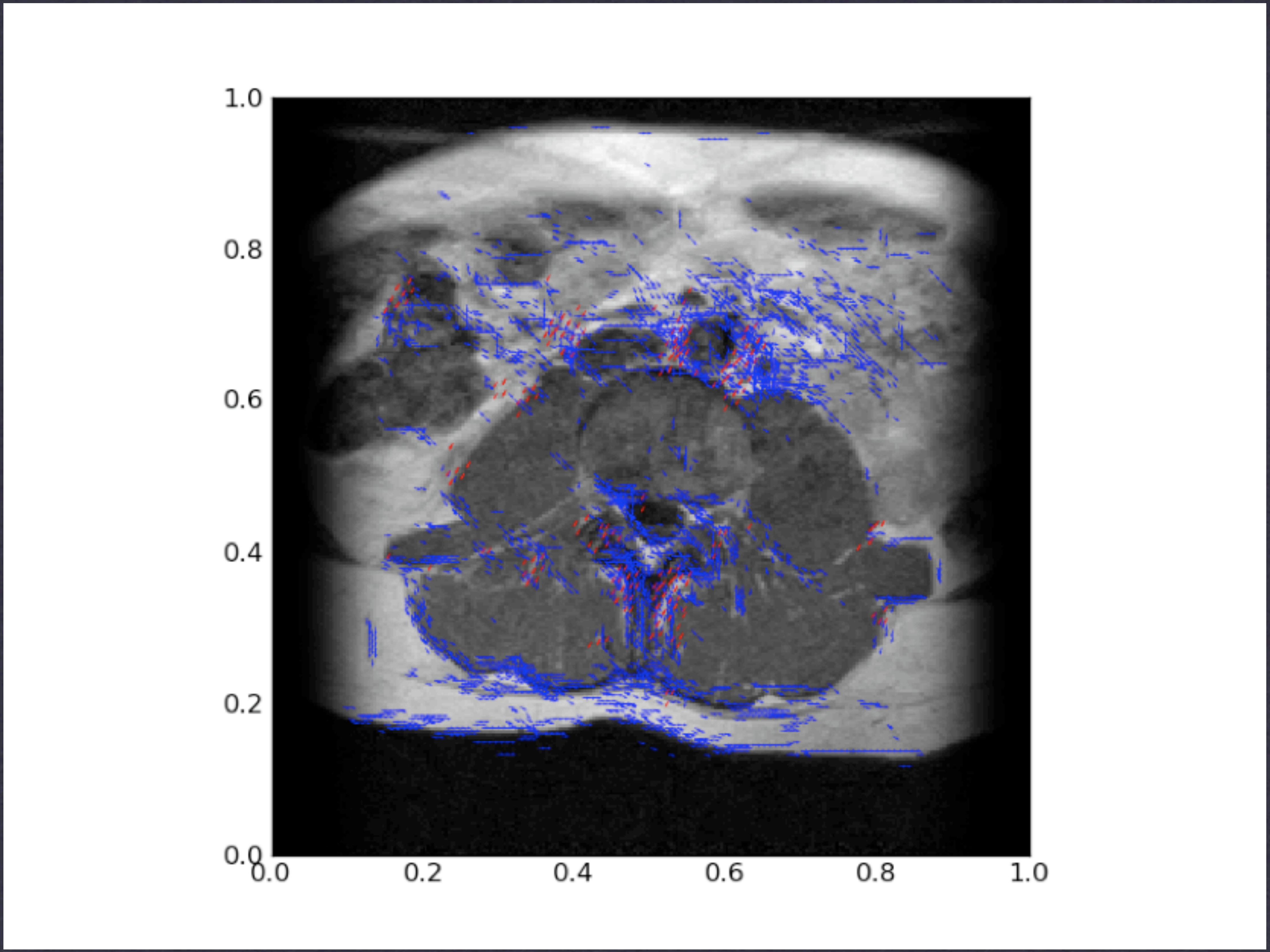


WAVEFRONT FILTERS

ARROWS ARE TANGENTIAL TO THE EDGE



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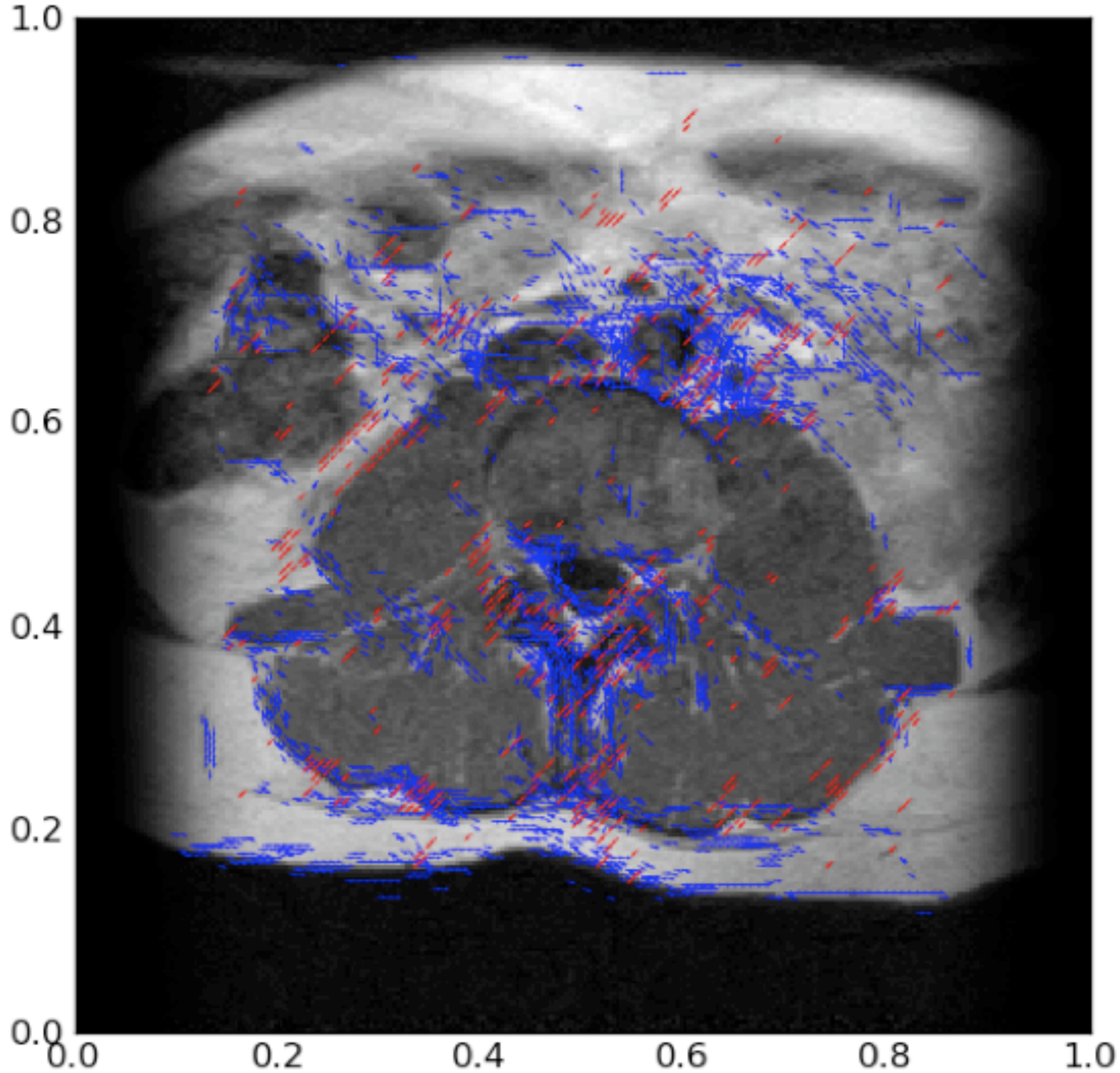

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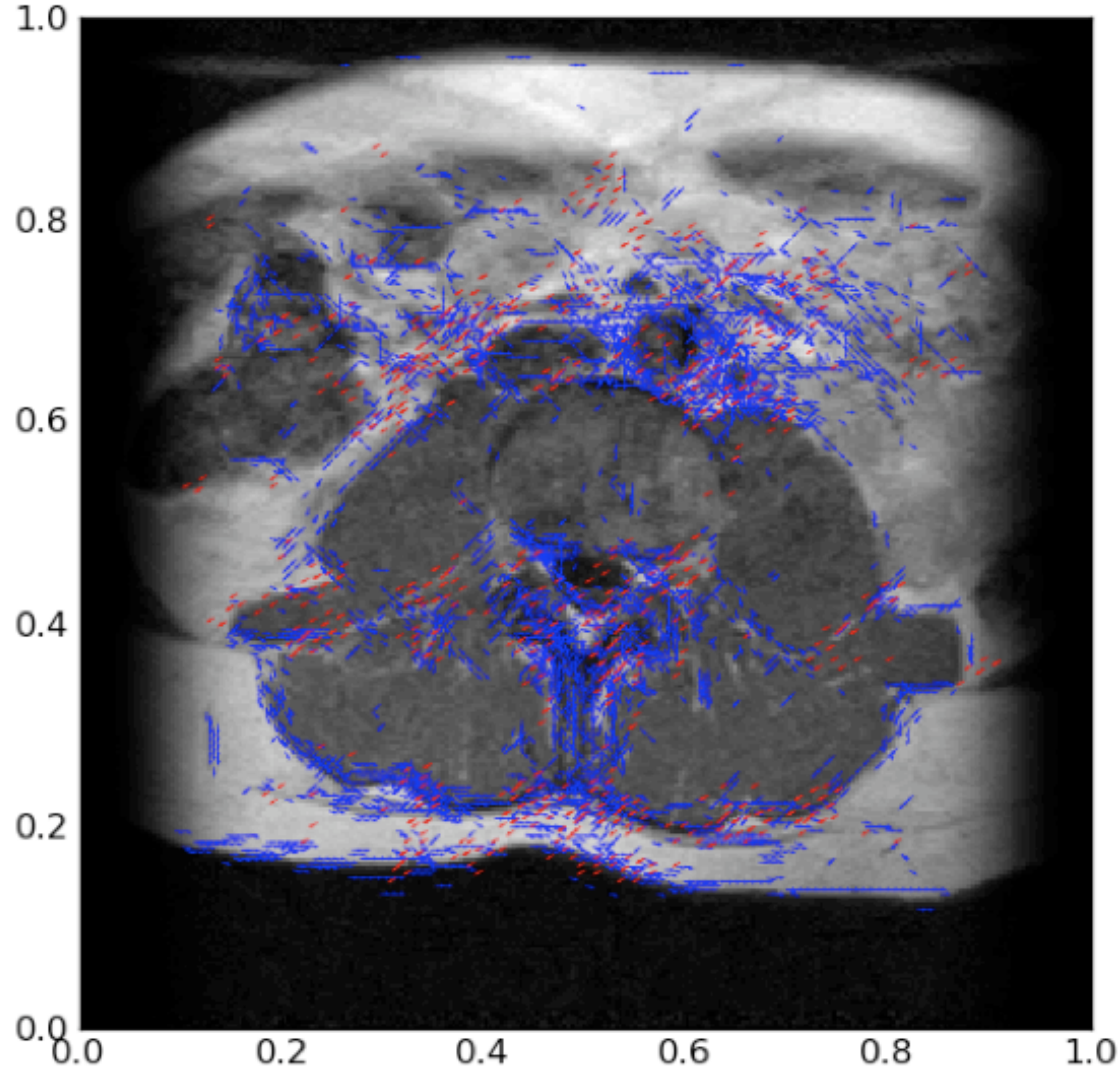

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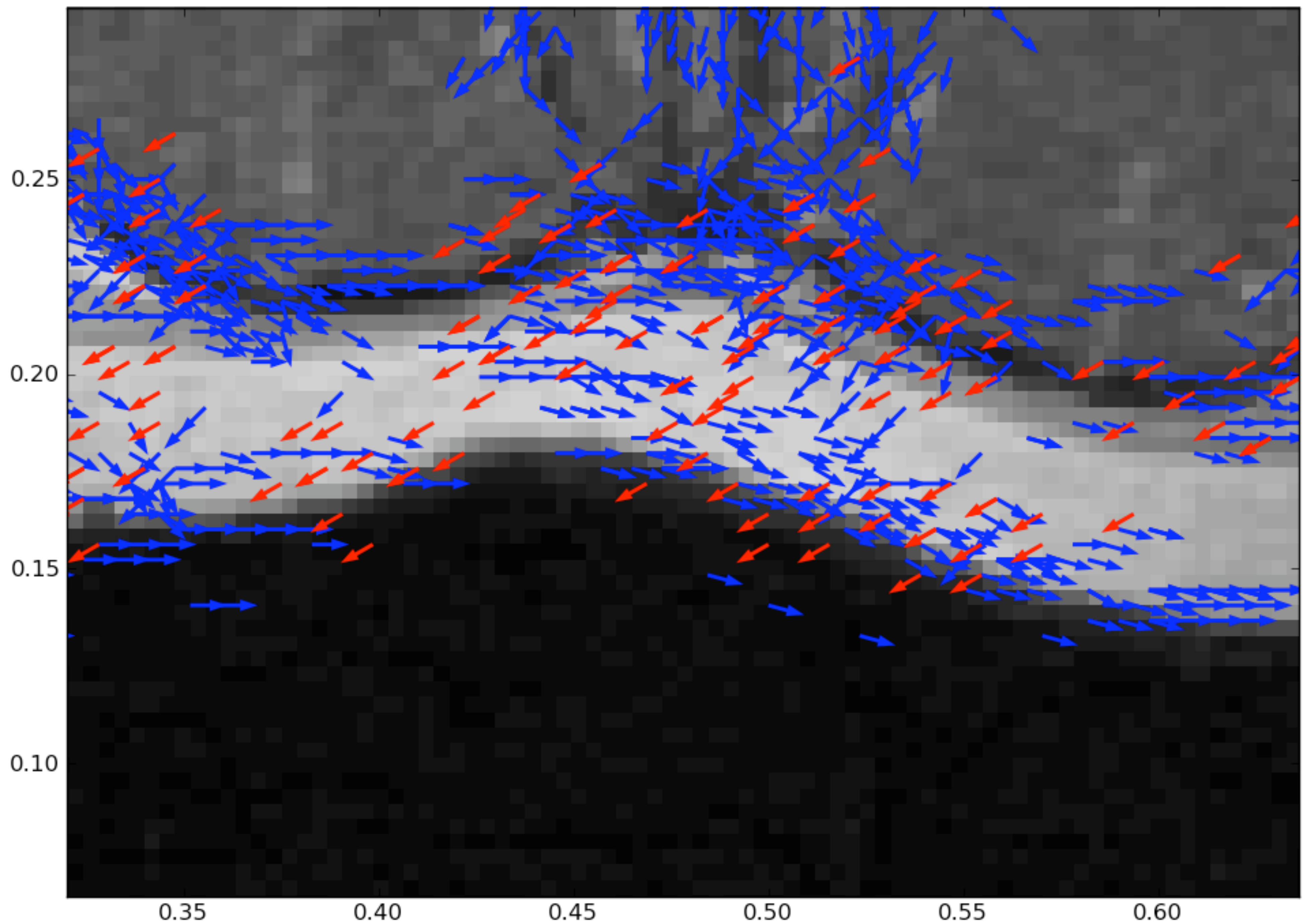
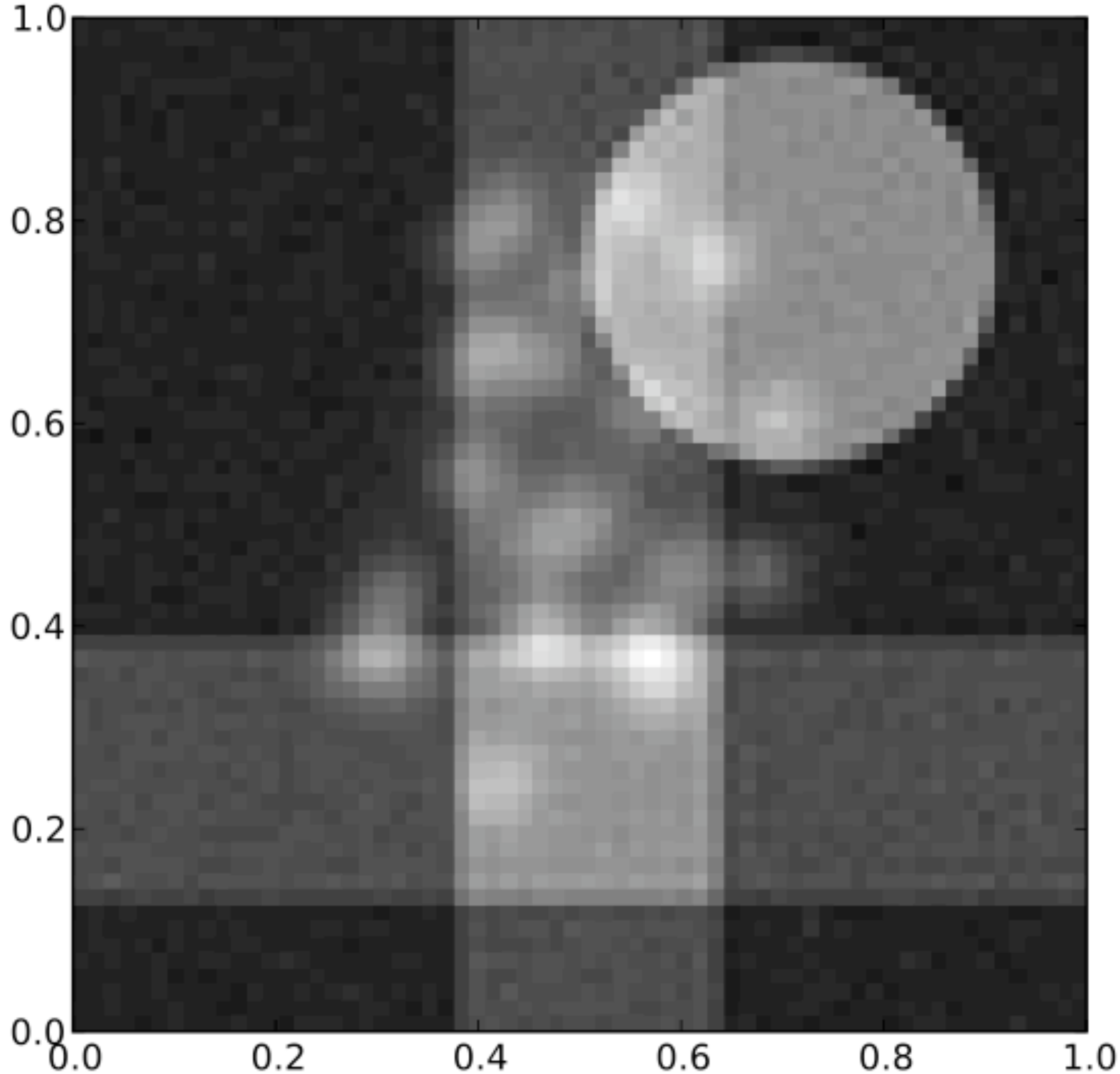
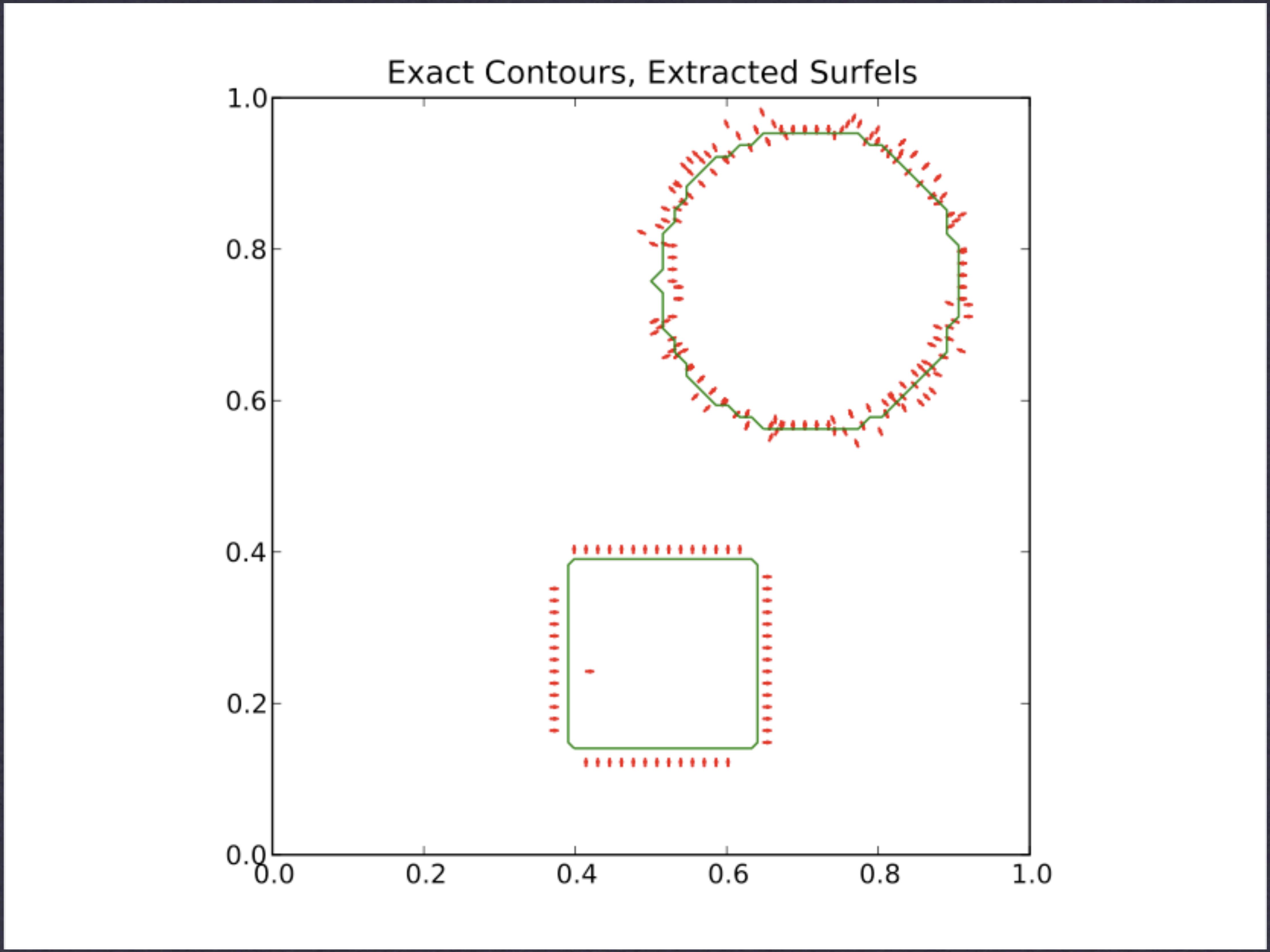


SKIN OF LOWER BACK



ZERO CURVATURE EXAMPLE

SPURIOUS EDGES ARE NOT DETECTED



ZERO CURVATURE EXAMPLE

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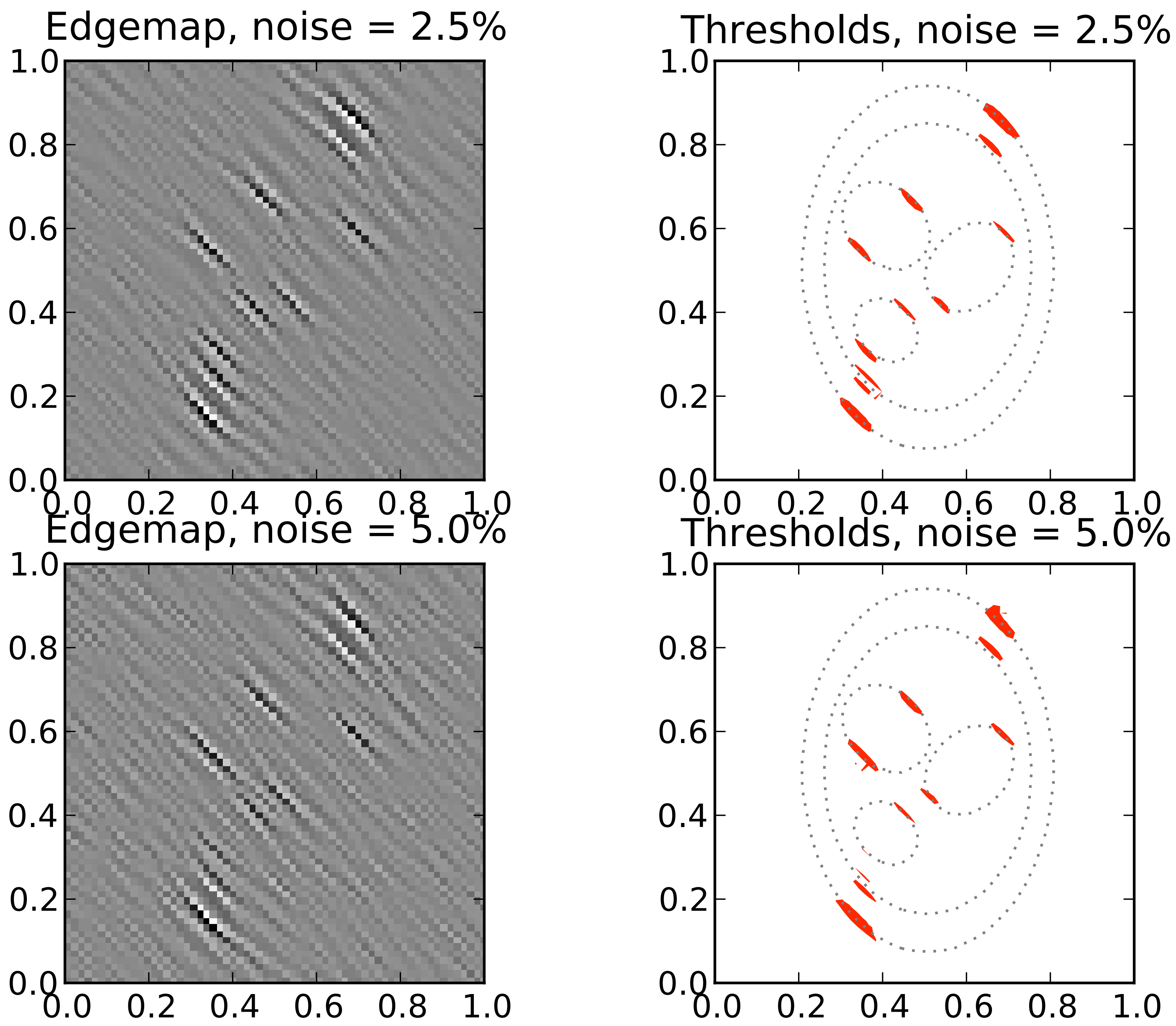
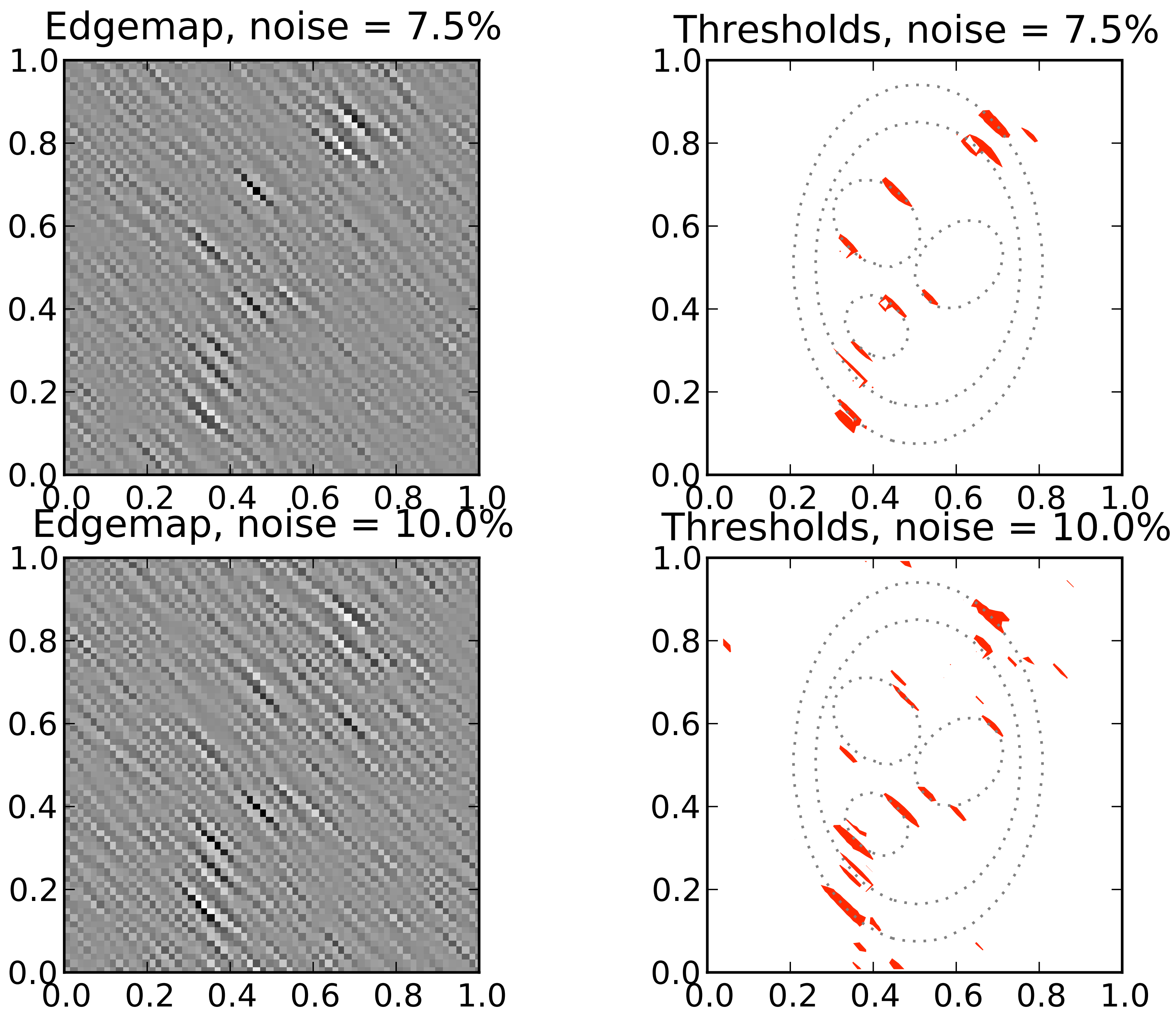


NOISE SENSITIVITY



NOISE SENSITIVITY

Analysis

Assumptions

- MRI measures Fourier transform of density
- Image piecewise constant plus smooth part
- The image boundaries are smooth
- Curvature bounded above and below
- The boundaries are separated from each other
- Minimum edge contrast

NOT SATISFIED IN PRACTICE

How it works

$$\frac{e^{ik\cdot\gamma_j(t_j(\vec{k}))}}{|k|^{3/2}}\frac{\sqrt{\pi}}{\sqrt{\kappa_j(t_j(\vec{k}))}} + O(|k|^{-5/2})$$

- Start with asymptotic expansion

How it works

$$\frac{e^{ik\cdot\gamma_j(t_j(\vec{k}))}}{|k|^{3/2}}\frac{\sqrt{\pi}}{\sqrt{\kappa_j(t_j(\vec{k}))}}\mathcal{V}(k_\theta)|k|^{1/2}\mathcal{W}(|k|)$$

- Drop higher order terms and apply directional filter

How it works

$$\int_{-\alpha}^{\alpha}\int_{0}^{\infty}\frac{e^{ik\cdot\gamma_j(t_j(k_\theta))}e^{-ik\cdot x}}{k_r^{3/2}}\frac{\sqrt{\pi}}{\sqrt{\kappa_j(t_j(k_\theta))}}\mathcal{V}(k_\theta k_r^{3/2}\mathcal{W}(k_r dk_r dk_\theta$$

- Then inverse Fourier Transform

How it works

$$\int_{t_j(-\alpha)}^{t_j(\alpha)} \int_0^\infty \frac{e^{ik\cdot(\gamma_j(t)-x)}}{k_r^{3/2}} \frac{\sqrt{\pi}}{\sqrt{\kappa_j(t)}} \mathcal{V}(k_\theta(t) k_r^{3/2} \mathcal{W}(k_r dk_r dt$$

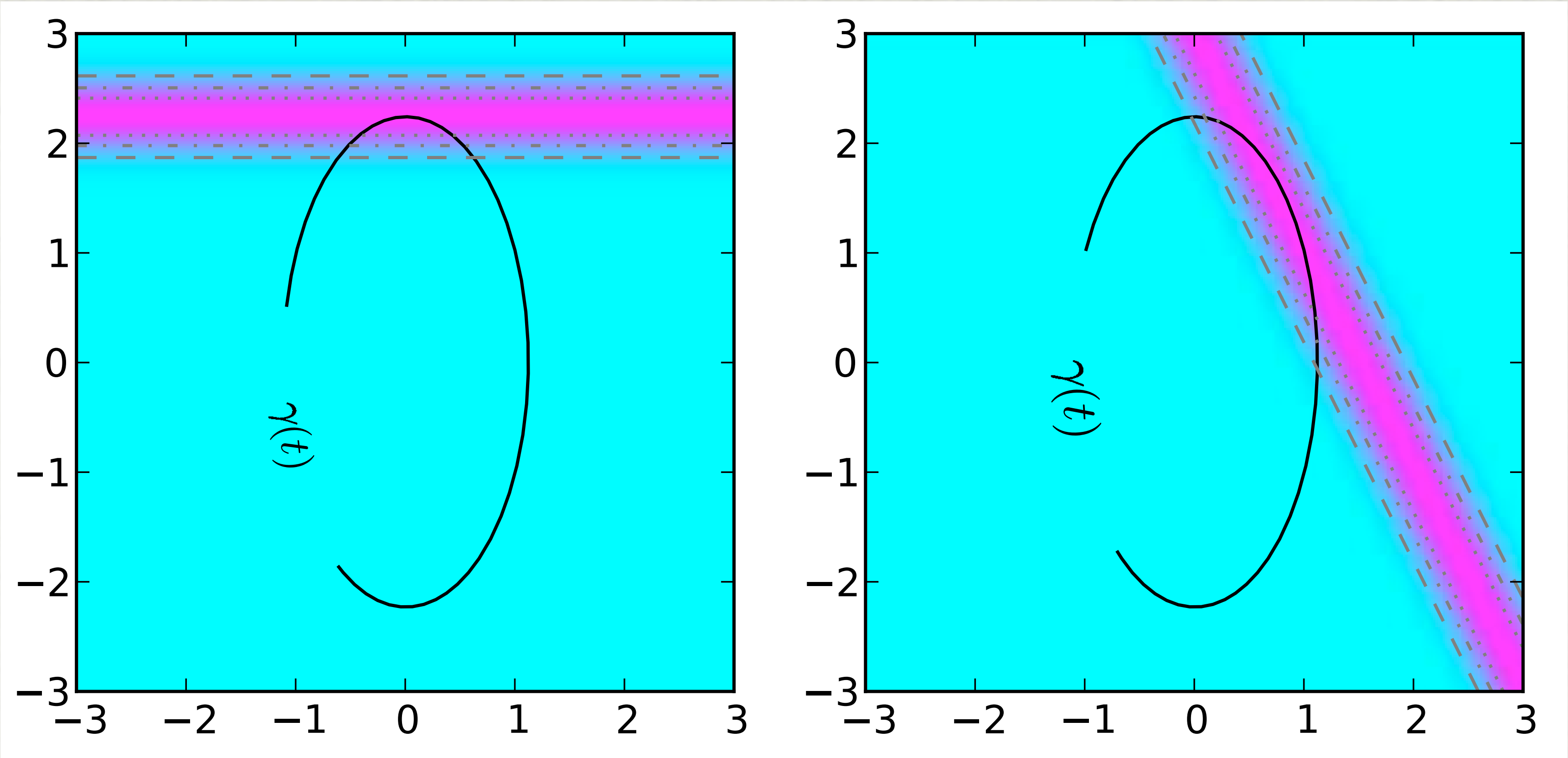
- Change variables

How it works

$$\int_{t_j(-\alpha)}^{t_j(\alpha)} e^{ik\cdot\gamma_j(t)} \sqrt{\pi\kappa_j(t)}\mathcal{V}(k_\theta(t)k_r^{3/2}\check{\mathcal{W}}(N_j(t)\cdot[\gamma_j(t)-x])dk_r dt$$

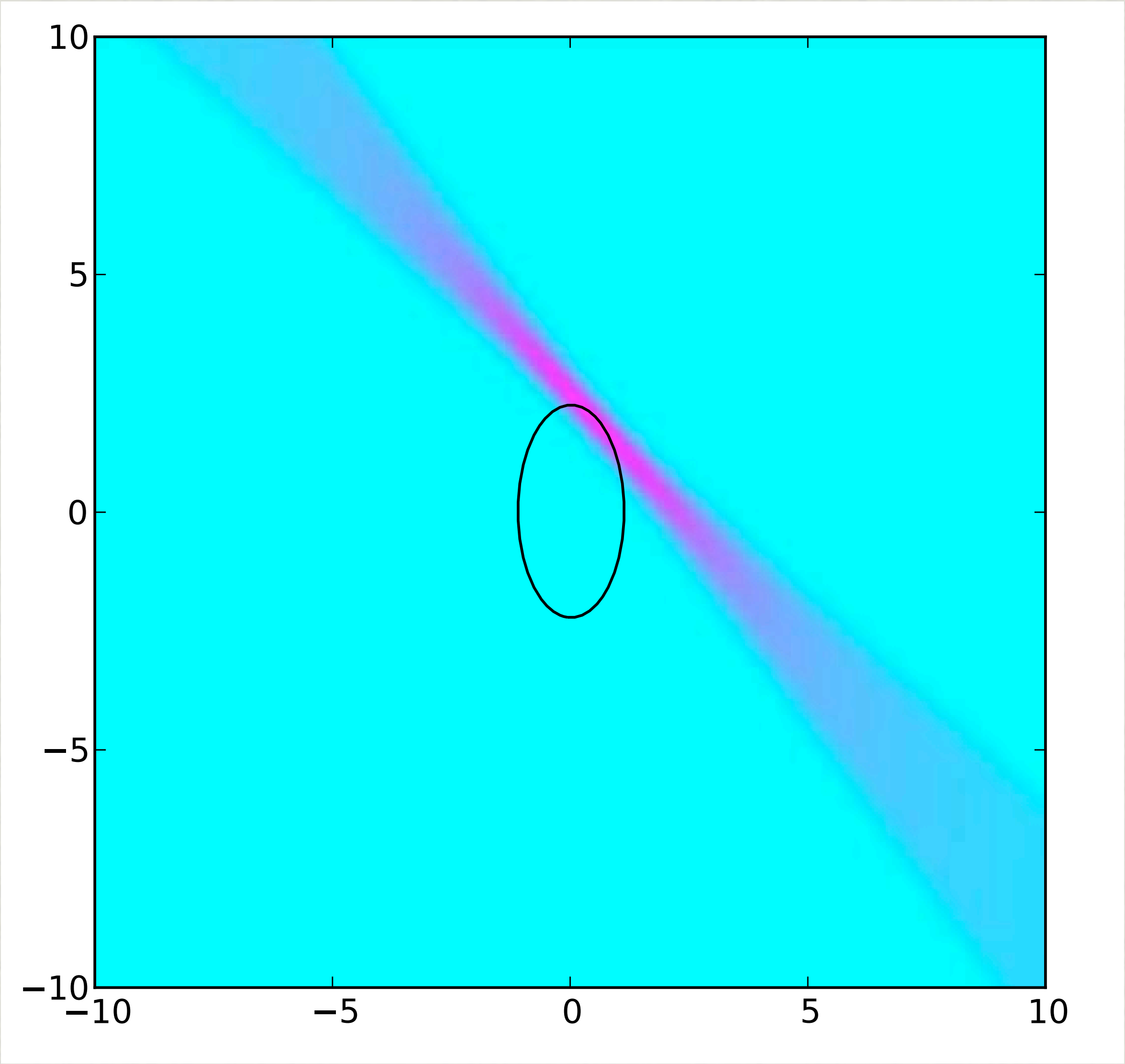
- And evaluate inner integral

Proof of Correctness



SCHEMATIC PLOT OF THE INTEGRAND

Proof of Correctness



- Fast decay in normal direction
- Polynomial decay in tangential direction
- Parabolic scaling:

$$\text{k domain:} \qquad \text{width} = O(\sqrt{\text{length}})$$

$$\text{x domain:} \qquad \text{width} = O(\text{length}^2)$$

Theorem

- A directional filter will extract at least one surfel near the point where the tangent of an edge equals the direction of the filter.
- It will not extract surfels far from the edge.
- The theorem only applies to unrealistic parameter choices. Algorithm still works on phantoms, however.

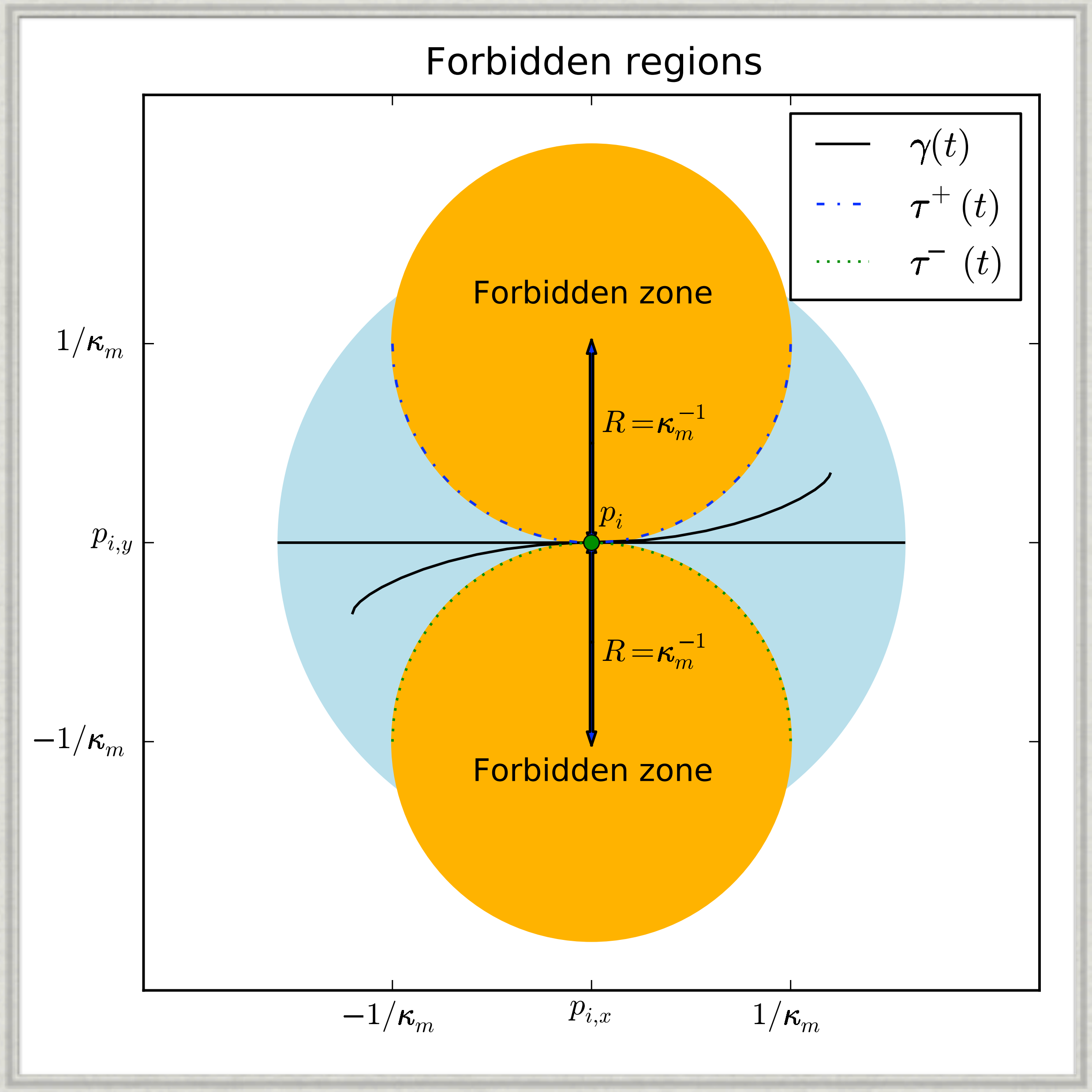
Segmentation with Surfels

Combinatorial Reconstruction

- Goal: combinatorial reconstruction of curves from scattered surfels
- How can tangential information help?

Combinatorial Reconstruction

- Points can only be connected in tangential direction.



Reconstruction Algorithm:

- Connect all points close to each other, but not within forbidden region.
- Prune graph, connecting only nearest tangential neighbors within the graph.
- Result is polygon with same topology as original curve.
- Then smooth polygon.

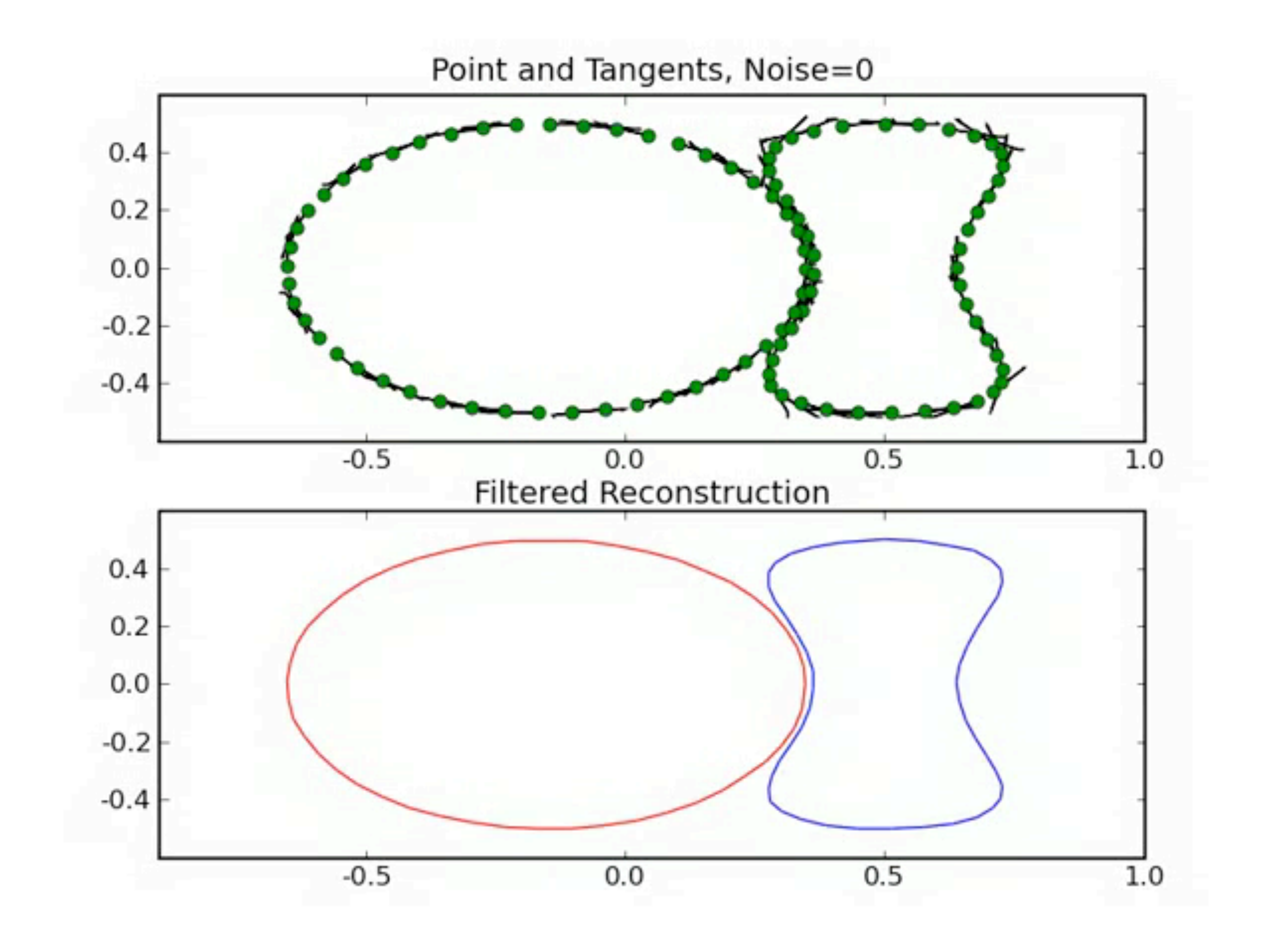
Reconstruction Algorithm

- Proven to work.

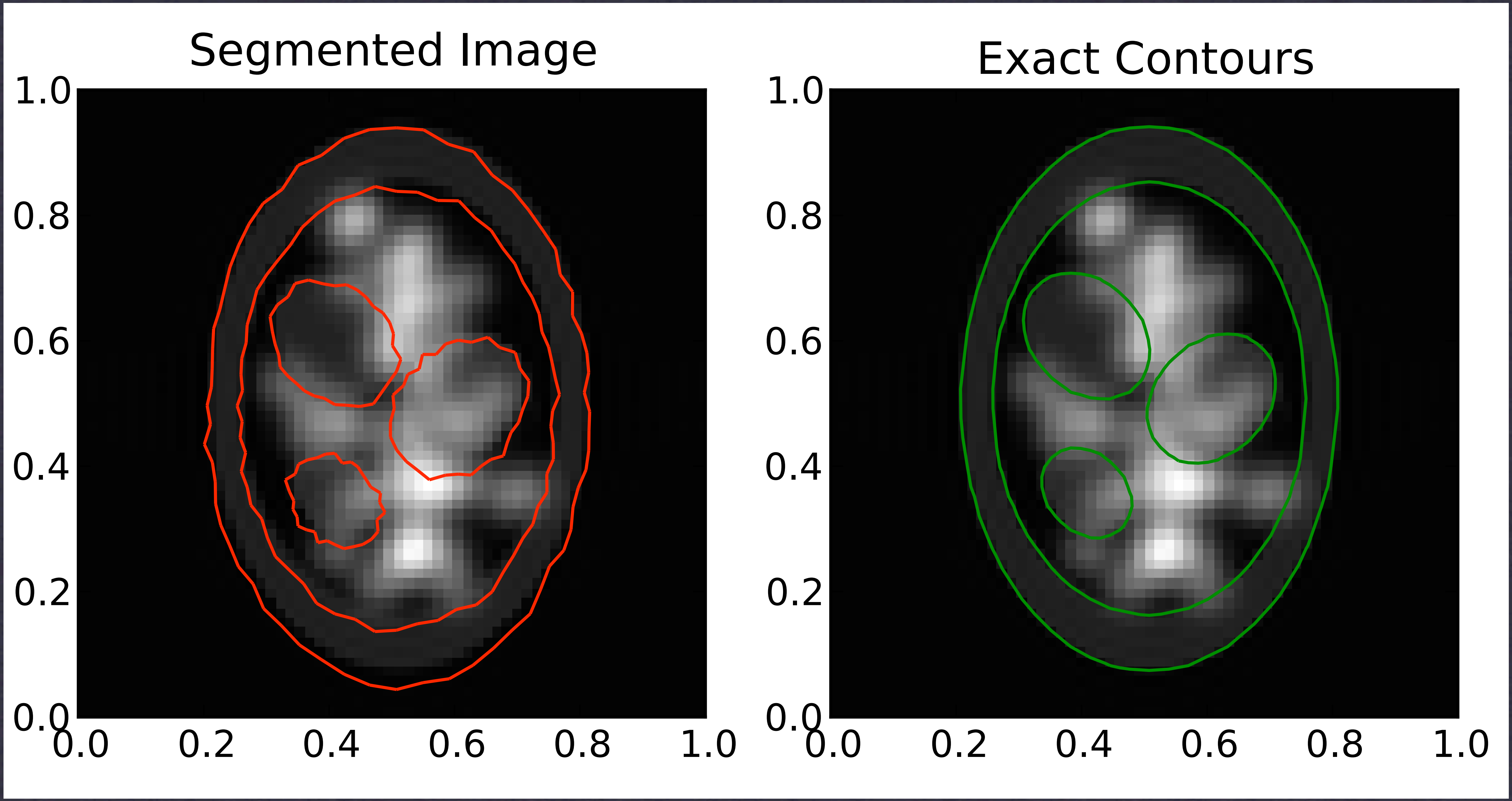
$$\text{sample spacing} = O(\sqrt{\text{curve separation}})$$

- Proof is an exercise in elementary calculus.
- Can filter *uncorrelated* noise via geometric constraints.

CURVE RECONSTRUCTION FROM POINTS AND TANGENTS.
L. GREENGARD AND C. STUCCHIO
ARXIV.ORG/ABS/0903.1817



FILTERING UNCORRELATED NOISE



SEGMENTED PHANTOM

OVERSIMPLIFIED GEOMETRY

Surfel Segmentation

- Can prove segmentation algorithm correct by plugging output of wavefront theorem into input of curve reconstruction theorem.
- Combinatorial curve reconstruction only works for simplified geometry.

Open problems

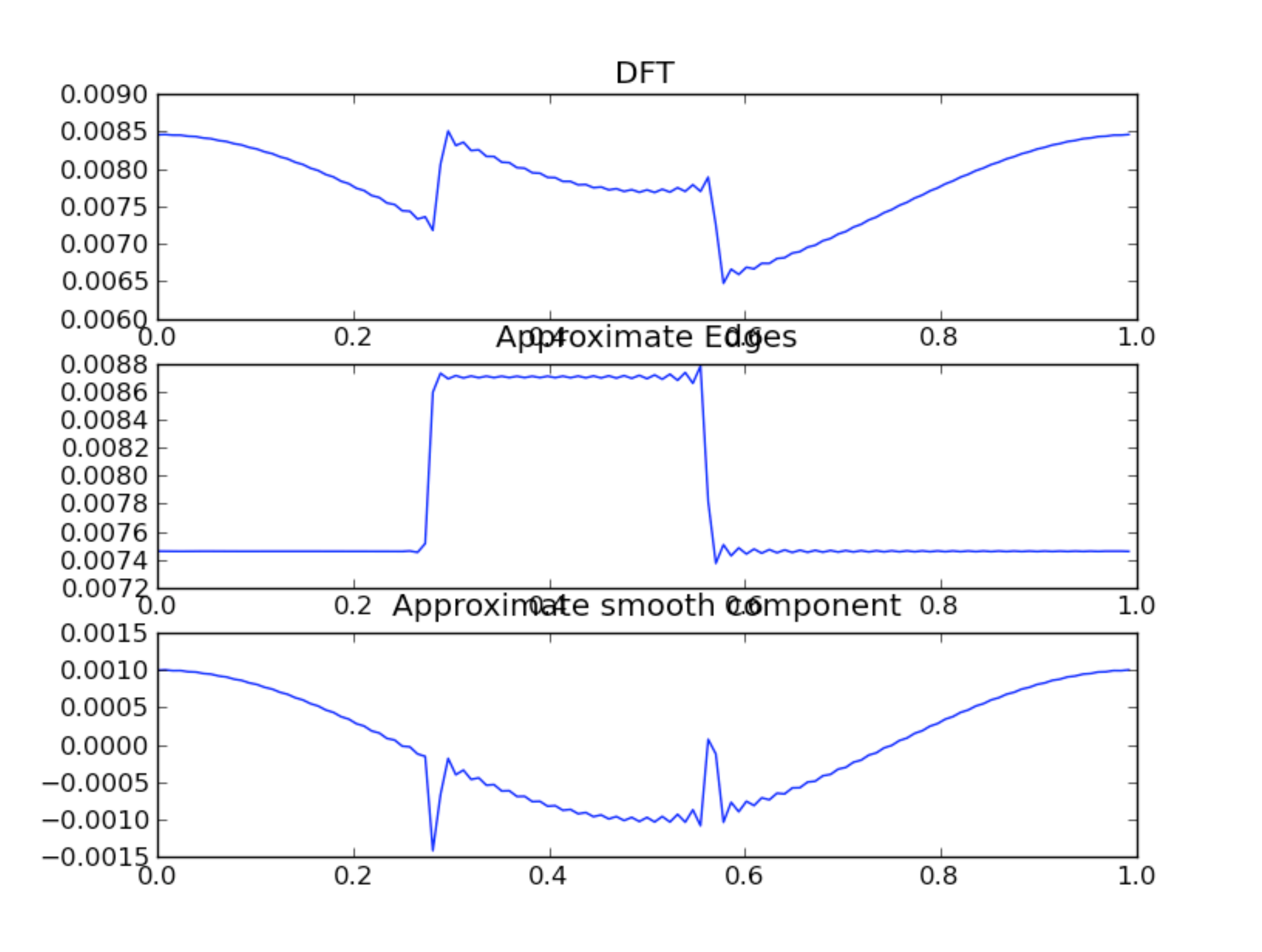
- Build a level-set based segmentation algorithm that uses surfel data.
- Clean up the surfel data (Bayesian tricks)

Reconstruction

Reconstruction

- *Assume* segmentation problem is approximately solved.
- Obvious idea: compute Fourier transform of discontinuities, subtract off, leaving only smooth part of function.
- Then manually draw discontinuities back.

Fail


DFT
0.0090
0.0085
0.0080
0.0075
0.0070
0.0065
0.0060
0.0
0.2
0.4
0.6
0.8
1.0
Approximate Edges
0.0088
0.0086
0.0084
0.0082
0.0080
0.0078
0.0076
0.0074
0.0072
Approximate smooth component
0.0015
0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.0015

Fourier Extension

- Best approximation to low frequency data:

$$\hat{\rho}_{\mathrm{meas}}(k)$$

- High frequency data missing, but we can approximate:

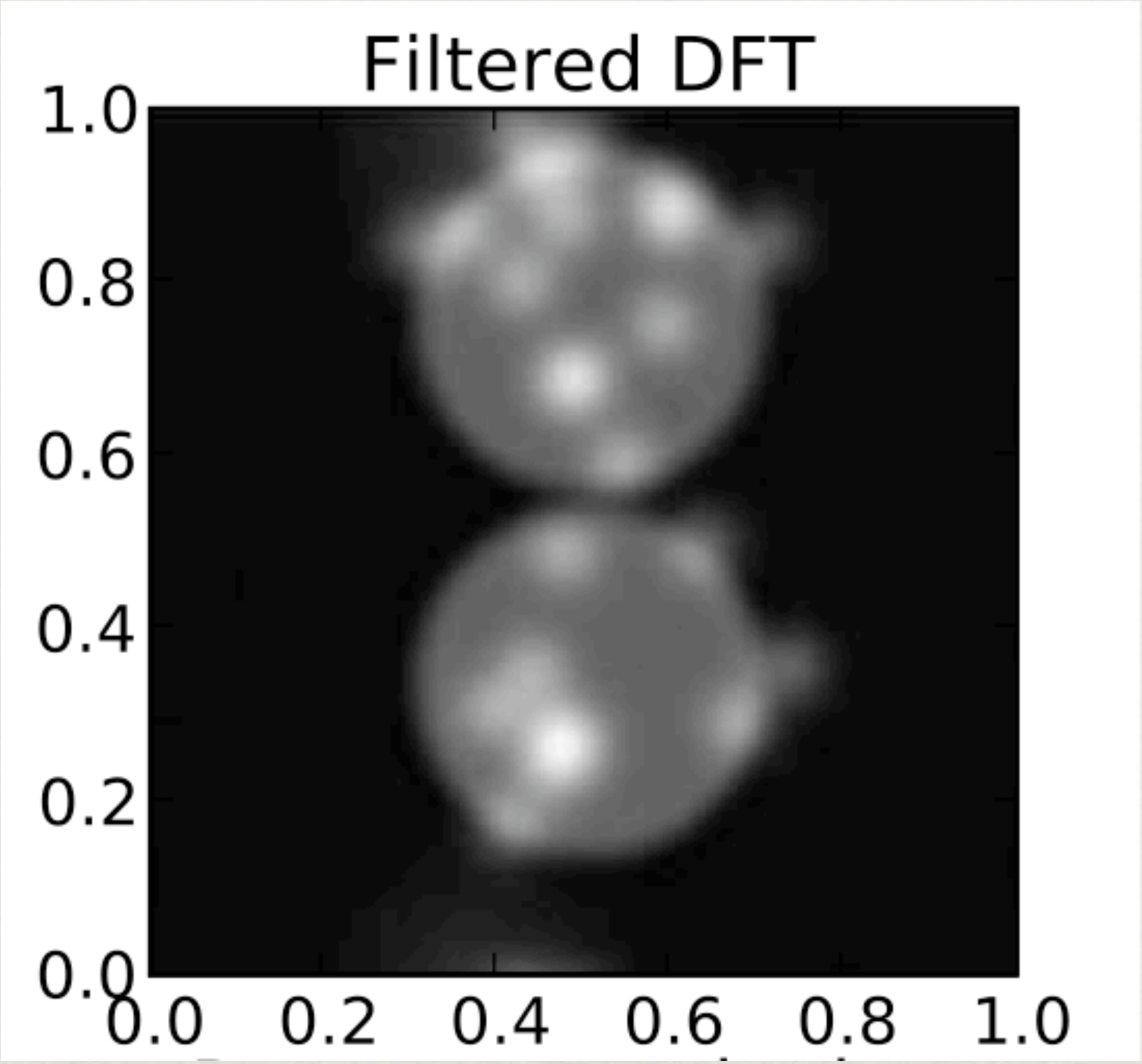
$$\sum_{j=1}^{M-1} \rho_j \widehat{1_{\gamma_j}}(k) = \sum_{j=1}^{M-1} \rho_j \frac{1}{i|k|^2} \int_{S^1} e^{ik\cdot\gamma_j(t)} k^{\perp} \cdot \gamma_j'(t) dt$$

Fourier Extension

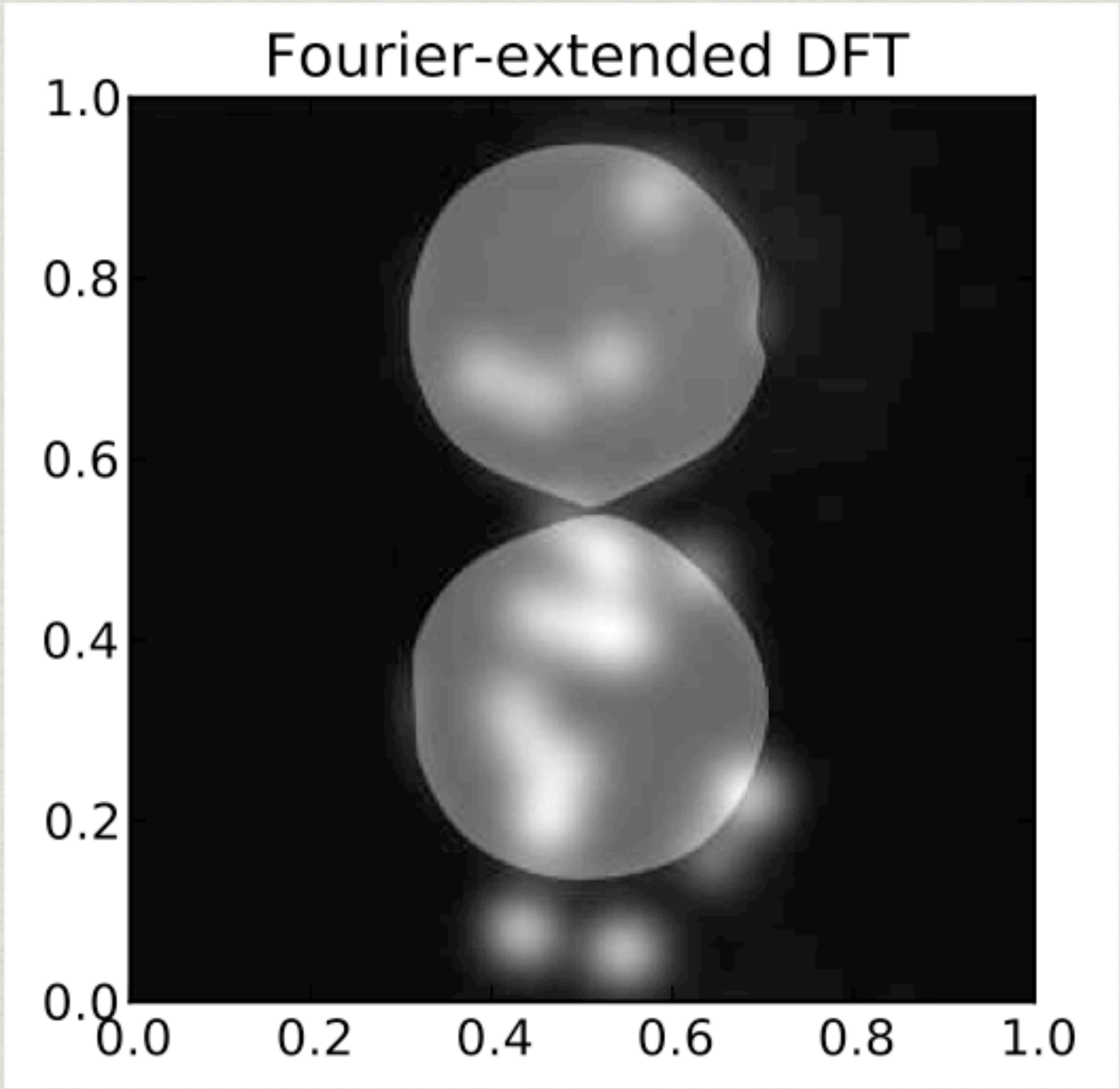
- Smooth Transition between them to avoid artifacts:

$$\begin{aligned}\hat{\rho}_{\text{reconstructed}}(k) &= \text{LPF}(k)\hat{\rho}_{\text{meas}}(k) \\ &+ \text{HPF}(k)\sum_{j=1}^{M-1}\rho_j\widehat{1_{\gamma_j}}(k)\end{aligned}$$

Fourier Extension



Fourier Extension



Conclusion

- The wavefront of an image has more information than it’s singular support
- Surfels can be extracted directly from raw data
- Effectively segments and reconstructs phantoms
- Still needed: good geometric algorithms for surfel reconstruction